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Dependence of the critical Richardson number on the temperature gradient in the mesosphere

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Abstract

Maximum upper atmospheric turbulence results in the mesosphere from convective and/or dynamic instabilities induced by gravity waves. For the first time, by comparing the vertical accelerations induced by wind shear and the buoyancy force, it is shown that the critical Richardson number Ri_c can be estimated. Dynamic instability is developed for $Ri < Ri_c$. This new approach, for the first time, makes it is possible to establish and estimate the temperature gradient impact on dynamic instability development. Regarding our results, Ri_c increases from 0.25 to 0.38 as the negative temperature vertical gradient increases from $\partial T/\partial z = 0$ to $\partial T/\partial z \leq -9$ K/km. However, Ri_c for the temperature, independent of altitude, is 0.25, coinciding exactly with the Ri_c commonly used and estimated in classical studies (Miles, 1961; Howard, 1961) and subsequent papers without the temperature impact. The increase in the Ri_c value strongly influences cooling, inducing the cooling rate increase. Also, our results show that criterion $Ri_c <$ 0.25 can only be used for the turbulent diffusion, which is characterized by eddies with sizes much smaller than the scale height of the atmosphere. The Ri_c value increases with the increasing size of the eddies, but the term "eddy diffusion" cannot be applied to transport due to the large-scale eddies (Vlasov and Kelley, 2015).

Key Words: 3334-Middle atmosphere dynamics, 3369-Thermospheric dynamics, 3379-Turbulence

1. Introduction

In general, the Richardson number Ri can be defined as the ratio of the destruction of turbulent kinetic energy by buoyancy forces due to the production of turbulent energy by the wind shear flow. This determination leads to the relation (see, for example, Peixoto and Oort (1992))

$$Ri = \omega_B^2 / S^2 , \tag{1}$$

where ω_B is the buoyancy frequency,

$$\omega_B^2 = \frac{g}{T}\left(\frac{\partial T}{\partial z} + g/C_p\right) , \tag{2}$$

and T is the temperature, g is the acceleration of gravity, C_p is the heat capacity at constant pressure, and

$$S = \frac{\partial V}{\partial z} \tag{3}$$

is the vertical shear of the horizontal wind with the velocity $V(z)$ height profile. It is generally accepted that a dynamic instability develops when the Richardson number is less than ¼, i.e., the parcel's vertical motion induced by wind shear dominates the motion induced by the buoyancy force. The former creates and the latter destroys these perturbations. Most authors use the critical Richardson number $Ri_c < 1/4$ without references. Some authors refer to Miles (1961) and Howard (1961). They consider the stable-stratified, horizontal shear flows of an ideal fluid. A set of studies takes into account the time-dependent shear flow and the results of laboratory experiments (Peixoto and Oort, 1992; Galperin et al., 2007). However, we could not find papers on the critical Richardson number that take the mesospheric conditions into account. Miles and other authors (Abarbanel et al., 1984; Ligniéres et al., 1999; Galperin et al., 2007) did not consider the

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temperature's influence on the Ri_c value. However, the eddy turbulence peak is observed in the mesosphere or the lower thermosphere where the large negative and positive gradients of the temperature occur. We could find just one paper [Hysell et al., 2012] on the estimate of the Ri_c value in the lower thermosphere. Using the data on observations of the sporadic *E* layer, Hysell et al. (2012) inferred the parameters of wind shear corresponding to the irregularities observed in the layer and estimated the Ri_c value of 0.75. However, the authors used the wrong formula for the background density, resulting in densities much larger than the observed atmospheric density corresponding to the hydrostatic equilibrium. It is shown in Appendix 3 how $0.7 < Ri_c < 0.8$ can be found due to the background density used by Hysell et al. (2012).

The principal measure of stability regarding the buoyancy effects of the density gradient overriding its inertial effects is the Richardson number given by formula (1) in Miles (1961), which can be written as

$$Ri = -g\frac{\partial\rho}{\partial z}/\left\{\rho\left[\frac{\partial V}{\partial z}\right]^2\right\}, \tag{4}$$

where ρ is the density and V is the horizontal wind velocity. This formula can be rewritten as

$$\left(\frac{\partial V}{\partial z}\right)^2 = -\frac{g}{Ri}\frac{1}{\rho}\frac{\partial\rho}{\partial z} . \tag{5}$$

This initial formula will be used here to estimate the accelerations induced by wind shear and the buoyancy forces under mesospheric conditions.

The goal of this paper is to estimate the critical Richardson number, Ri_c, corresponding to the equilibrium between the buoyancy force and the force induced by wind shear in the mesosphere.

Dynamic instability is developed for $Ri < Ri_c$. Our approach considers the acceleration corresponding to both forces, taking into account the mesospheric temperature height distributions.

2. Acceleration Induced by Wind Shear

We start from formula (5) corresponding to the initial equation used by Miles (1961) (here, formula (4)). Miles considers an uncompressible fluid but the adiabatic expansion/compression should be taken into account in the upper atmosphere. Differentiating the adiabatic relation $pT^{-\gamma/(\gamma-1)} = const$ corresponding to Poisson's equation where $p = \rho\kappa T/m$ and p is the pressure; m is the mean molecular mass; $\gamma = C_p/C_v$; C_p and C_v are the heat capacities at constant pressure and volume; $\gamma/(\gamma-1) = 1 + N/2$; $N = 5$ is the number of degrees of freedom for diatomic gas; and κ is the Boltzmann's constant, it is possible to get the adiabatic expansion equation

$$\frac{1}{\rho}\frac{\partial\rho}{\partial z} = \frac{N}{2}\frac{1}{T}\frac{\partial T}{\partial z} \tag{6}$$

(see the derivation of this formula in Appendix1), and according to formula (5):

$$\left(\frac{\partial V}{\partial z}\right)^2 = -\frac{g}{Ri}\frac{N}{2T}\frac{\partial T}{\partial z} \ . \tag{7}$$

Taking into account $Ri(\partial V/\partial z)^2 = \omega_B^2 = (g/T)\left(\partial T/\partial z + g/C_p\right)$ and using formula (6), the temperature gradient in the parcel with upward motion and adiabatic expansion can be given by the equation

$$\frac{\partial T}{\partial z} = -\frac{g}{(1+N/2)C_p} \tag{8}$$

and

$$T = T_0 - \frac{g}{\left(1+\frac{N}{2}\right)C_p}(z - z_0) \ . \tag{9}$$

By substituting formulas (8) and (9) in formula (7) multiplied by $(z - z_0)$, it is possible to obtain the formula

$$a_{ws} = \frac{g^2 N(z-z_0)}{2Ri[T_0 C_p(1+N/2) - g(z-z_0)]} \tag{10}$$

where

$$a_{ws} = \left(\frac{\partial V}{\partial z}\right)^2 (z - z_0) \tag{11}$$

is the acceleration in wind shear. As can be seen from Fig. 1, this acceleration increases with the increase of the vertical size of the wind shear layer. Note that this size cannot exceed 1–2 km according to the experimental data (Larsen, 2002). The a_{ws} dependence on the altitude is linear because $g(z - z_0) \ll T_0 C_p (1 + N/2)$ for $-z_0 < 2$ km.

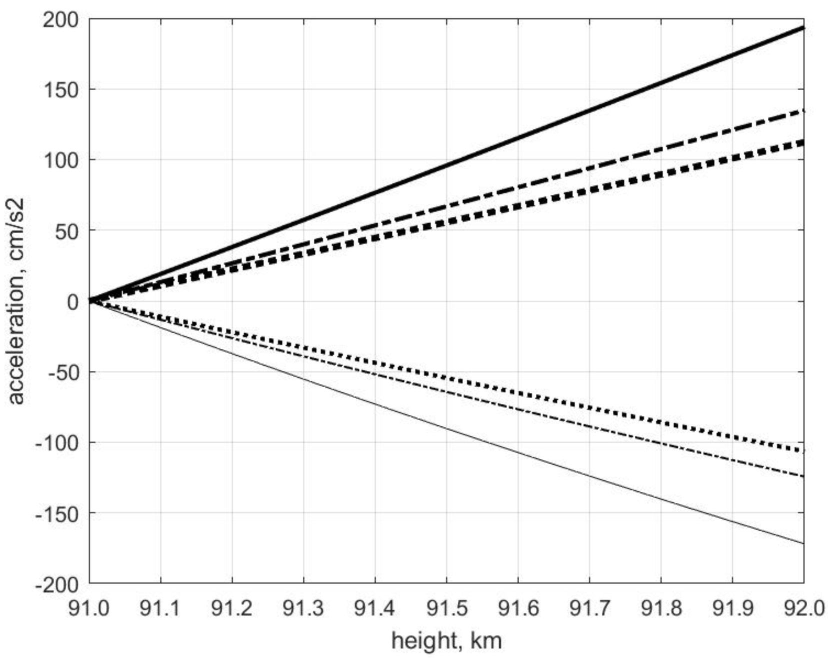

Figure 1. The height profiles of the wind shear $a_{ws} > 0$ and buoyant $a_B < 0$ accelerations calculated by formulas (11) and (15), respectively, with T_0 = 140 K and Ri_c = 0.25 (solid curves), with T_0 = 140 K and $Ri_c = R/C_p$= 0.286 (dashed-dotted curves), and with T_0 = 200 K and Ri_c = 0.286 (dotted curves).

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3. Acceleration Induced by the Buoyancy Force

The buoyancy force is $F_B = g(\rho_A - \rho_D)$ where ρ_A and ρ_D are the background atmospheric density and the disturbed density, respectively. The acceleration is given by

$$a_B = g[(\rho_A - \rho_D)\rho_D]\ . \tag{12}$$

The atmospheric density distribution can be given by

$$\rho_A = \rho_{A0} exp[-(z - z_0)/H_A] \tag{13a}$$

for $dT_A/dz = 0$ in the mesopause and the formula

$$\rho_A = \rho_{A0}\{[T_{A0} - G(z - z_0\ )\ ]/T_{A0}\}^{(mg/\kappa G - 1)} \tag{13b}$$

for $dT_A/dz = G < 0$ below the mesopause, and $H_A = \kappa T_{A0}/\text{mg}$ is the scale height of the atmospheric gas. By integrating equation (6) with the temperature and temperature gradient given by formulas (8) and (9)**,** it is possible to get the disturbed density distribution ($T_0 = T_{A0}$),

$$\rho_D = \rho_{A0}\left[\frac{T_0 - \frac{G(z-z_0\ )}{C_p(1+N/2)}}{T_0}\right]^{N/2}\ , \tag{14}$$

and the acceleration corresponding to the buoyancy force can be written as

$$a_{\boldsymbol{B}} = g\left[\left(\frac{\rho_A}{\rho_D}\right) - 1\right] = g\,\frac{\rho_{A0} e^{-\frac{(z-z_0)}{H_A}}}{\rho_{A0}\left[\frac{T_0 - \frac{g(z-z_0\ )}{C_p\left(1+\frac{N}{2}\right)}}{T_0}\right]^{\frac{N}{2}}} - g \tag{15}$$

for $dT_A/dz = 0$. As seen from Fig. 1, there is very good agreement between the a_{ws} and a_B absolute values for Ri_c = 0.25, and T_0 = 140 K and T_0 = 200 K for the vertical size of a stable wind shear layer that is less than 400 m. The a_{ws} value becomes larger than the a_B value for $z - z_0 >$ 400 m, which means that the Ri_c value should be increased. The turbulence develops if α_{ws} is larger than the α_B that corresponds to $Ri < Ri_c$. We emphasize that the perturbation scale sizes

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induced by wind shear do not exceed 1-2 km, according to the observations (see Lübken (1997)). Note that formula (13b) should be used instead of formula (13a) in the nominator of formula (15) for atmospheric temperature distribution with $\frac{dT_A}{dz} < 0$. As can be seen from Fig. 2, the a_B values significantly decrease in this case, since the atmospheric density given by formula (13b) is larger and the density gradient is less than the density and gradient corresponding to formula (13a). The small buoyancy force corresponds to the small density gradient. This dependence explains the a_B reduction with the T_A decrease.

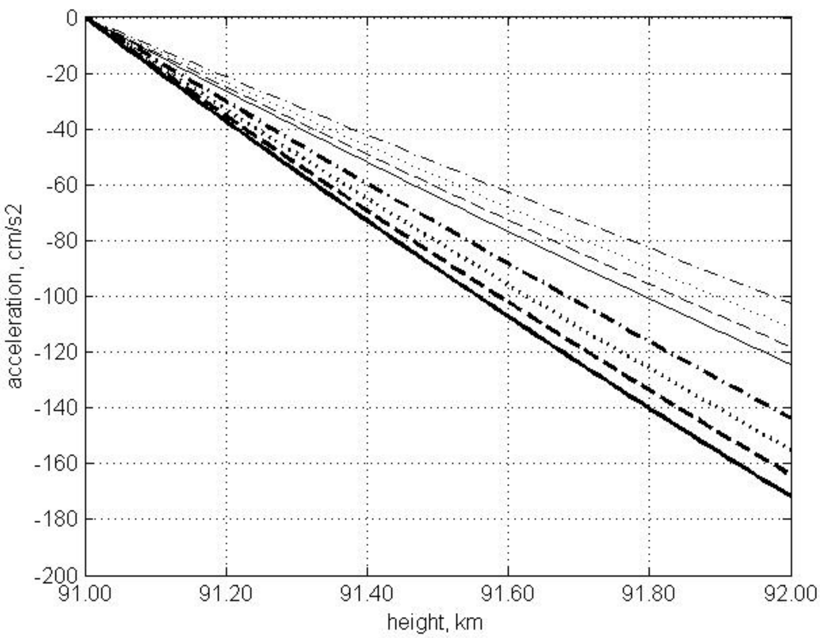

Figure 2. The height profiles of the acceleration of the buoyancy force calculated by formula (15) with the nominator $\rho_{A0}\{[T_{A0} - G(z - z_0)]/T_{A0}\}^{(mg/\kappa G - 1)}$ for $T_0 = T_{A0}$ = 140 K and 200 K (thick and thin curves, respectively) and G = 1, 2.8, and 5 K/km (dashed, dotted and dashed-dotted curves, respectively), and calculated by formula (15) (solid curves).

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4. Estimating the Richardson Number

Using formulas (11) and (15) in the equation $a_{ws} + a_B = 0$, the formula for Ri_c can be inferred:

$$Ri_c = \frac{\left[1-\frac{g(z-z_0)}{T_0 C_p(1+N/2)}\right]^{N/2} \frac{gN(z-z_0)}{2C_p(1+N/2)\left[T_0-\frac{g(z-z_0)}{C_p(1+N/2)}\right]}}{\left[1-\frac{g(z-z_0)}{T_0 C_p(1+N/2)}\right]^{N/2} - exp\left[-\frac{(z-z_0)}{H_A}\right]} \, . \qquad (16)$$

The Ri_c values calculated by formula (16) and this formula with $\{[T_0 - G(z-z_0)]/T_0\}^{(mg/\kappa G-1)}$ (see formula (13b) instead of the exponential term) are shown in Figs. 3a and 3b. The Ri_c values increase with increasing altitude, corresponding to the vertical expansion of the region of the stable wind shear. However, according to the experimental data (Larsen, 2002; Kelley et al., 2003; Bishop et al., 2004), the wind shears are very unstable. As mentioned above, the size scales of the density perturbations do not exceed 1 – 2 km, according to the observations. A more accurate consideration of eddy turbulence (Vlasov and Kelley, 2015) concludes that the scale size of density perturbations l should be much less than the scale height of atmospheric gas, $l << H_A$ and $l << 4$ km for $T_A = T_0 = 140$ K and $l << 5.7$ km for $T_A = T_0 = 200$ K. However, this restriction can only apply to turbulence corresponding to the eddy diffusion approximation (Vlasov and Kelley, 2015). As seen from Fig. 3a, the Ri_c value of 0.25 corresponds to perturbations with scales less than 10 m, and the Ri_c values reach 0.256 and 0.263 for l = 200 m and 400 m and for $T_{A0} = 140$ K and 0.254 and 0.257 for $T_0 = 200$ K, respectively. The Ri_c value of 0.25 corresponds to the mean value l = 27.3 m obtained by Lübkin (1997), using the measured spectrum of the density fluctuation. Vlasov and Kelley (2015) reconsidered the results of Kelley et al. (2003) and found that the spectrum scale fluctuations inferred from the meteor train turbulence observations can be approximated by Heisenberg's formula with l = 119 m, and eddies with very large scales may

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occur in the narrow layer of localized turbulence. As can be seen from Fig. 3b, the Ri_c values increase with the increase in the negative gradient of the temperature and can reach almost 0.36.

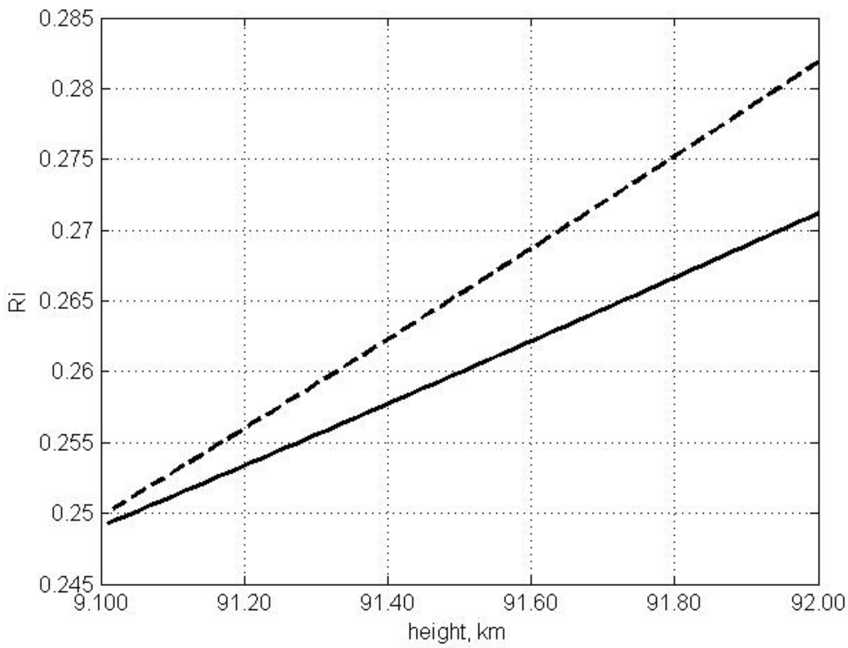

Figure 3a. The height profiles of the critical Richardson number calculated by formula (16) with T_0 = 140 K and 200 K (dashed and solid lines, respectively).

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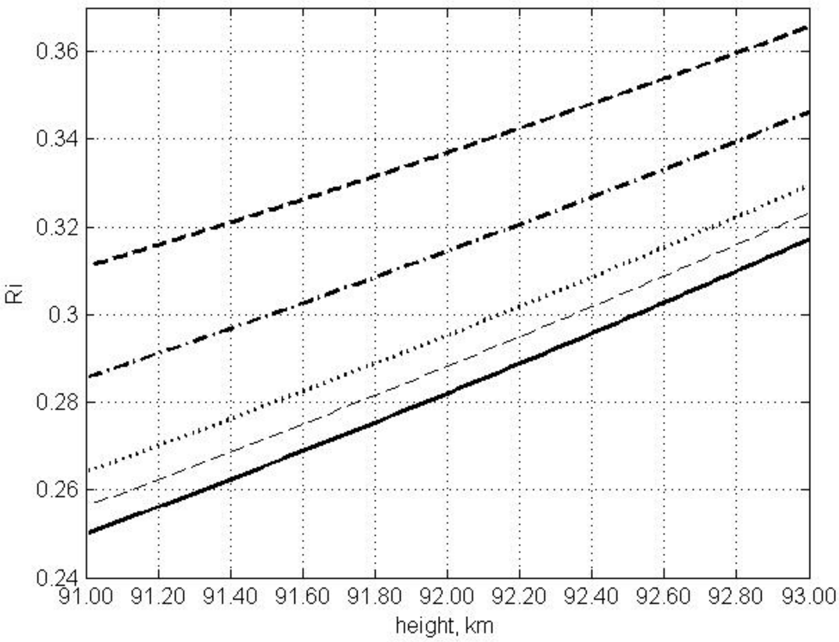

Figure 3b. The height profiles of the critical Richardson number calculated by formula (16) with $\{[T_0 - G(z - z_0)]/T_0\}^{(mg/\kappa G - 1)}$ instead of the exponential term for the T_0 = 140 K with $dT/dz =$ $G < 0$ with $|G|$ = 0.2, 1, 3, and 5 K/km (dashed thin, dotted, dashed-dotted and dashed thick curves, respectively) and calculated by formula (16) (solid thick curve).

Thus, turbulence can develop with Ri_c > 0.25 for wind shears with a vertical size of 1–2 km, but this turbulence may not correspond to eddy diffusion. The scales of the density fluctuations are very small (for example, see Lübken (1997)) that correspond to $z \to z_0$. However, the Ri_c value estimation for $z \to z_0$ is problematic because, in this case, the numerator and denominator in formula (16) try to attain zero. This uncertainty can be solved using L'Hospital's rule, leading to the formula (see Appendix 2)

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$$Ri_c = \frac{0.5gN}{g(1+N/2)^2 - 0.5gN - GC_p(1+N/2)} \quad (17)$$

for the Ri_c limit value for $z \to z_0$. This formula corresponds to the limit value formula (16) with the term $\{[T_0 - G(z - z_0)]/T_0\}^{(mg/\kappa G - 1)}$ instead of the term $exp[-(z - z_0)/H_A]$. The Ri_c dependence on the negative temperature gradient, given by formula (17), is shown in Fig. 4. The G increase improves the conditions for the dynamic instability development. Note that the Ri_c value for G = 0 coincides with the results of Miles (1961) and the commonly used value of Ri_c.

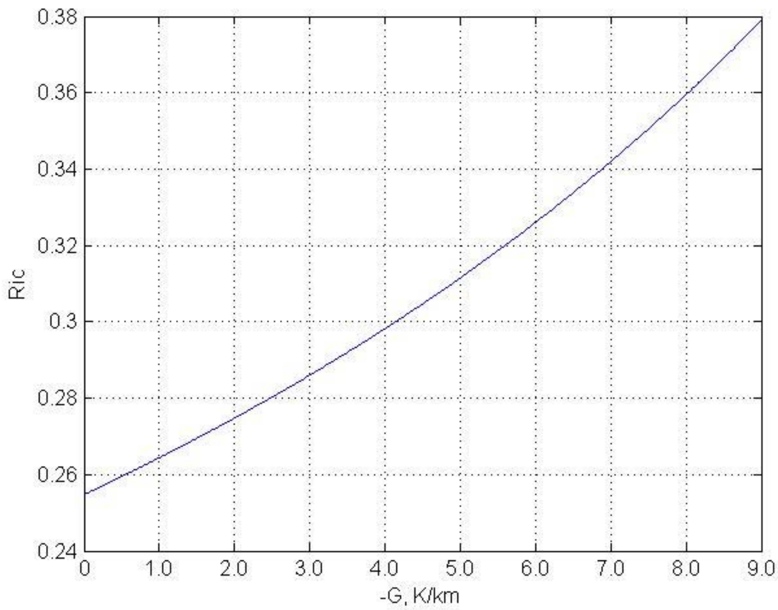

Figure 4. The dependence of the Richardson number Ri_c on the temperature negative gradient calculated by formula (17).

5. The Influence of Ri_c Dependence on G on Cooling in the Mesosphere

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The eddy turbulence heating/cooling rate can be given by the equation (Vlasov and Kelley, 2010)

$$Q_{ed} = \frac{\partial}{\partial z}\left[K_{eh} C_p \rho \left(\frac{\partial T}{\partial z} + \frac{g}{C_p}\right)\right] + K_{eh} \rho \frac{g}{Tb}\left(\frac{\partial T}{\partial z} + \frac{g}{C_p}\right), \qquad (18)$$

where K_{eh} is the coefficient of the eddy heat transport, ρ is the undisturbed gas density, and b is a dimensionless constant given by the relation obtained using the results of Gordiets et al. (1982),

$$b = Ri_c/(P - Ri_c) \qquad (19)$$

where P is the turbulent Prandtl number. According to equation (18), the Q_{ed} value is given in units $erg \times cm^{-3} \times s^{-1}$. The K_{eh} value is given by

$$K_{eh} = b\varepsilon/\omega_B^2, \qquad (20)$$

where ε is the energy dissipation rate, and b can be given by formula (19). The vertical distribution of the ε value in the turbulent layer can be approximated by the Gaussian function

$$\varepsilon = \varepsilon_m exp[-(z - z_m)^2/h^2], \qquad (21)$$

where h is half of the layer thickness and ε_m is the ε value at the altitude of the layer peak z_m. Using this approximation, dividing equation (18) by ρC_p and substituting formula (20) with $b = Ri/(P - Ri)$ and $T = T_0 + G(z - z_0)$, equation (18) can be written in units K/s as

$$Q_{ed} = \varepsilon_m exp\left[-\frac{(z-z_m)^2}{h^2}\right]\left\{\frac{[T_0+G(z-z_0)]}{g\left(\frac{P}{Ri_c}-1\right)}\left[-\frac{2(z-z_m)}{h^2} - \frac{\frac{mg}{\kappa}}{T_0+G(z-z_0)}\right] + \frac{1}{C_p}\right\}. \qquad (22)$$

Using the Ri_c dependence on the temperature gradient given by formula (17), the impact of the Richardson number on the cooling rates can be estimated. According to the results in Fig. 5, the cooling rates increase by a factor of 2.2 for $0.25 < Ri_c < 0.38$ corresponding to $0 \leq G \leq$ -9 K/km, but the G value influence on the cooling for Ri_c = const = 0.25 is very small (curves near the thick solid curve). Note that the turbulence induced by the large wind shear may not correspond to the

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eddy diffusion heat transport. The values of ε_m, z_m, and h correspond to the experimental data (Lübken, 1997).

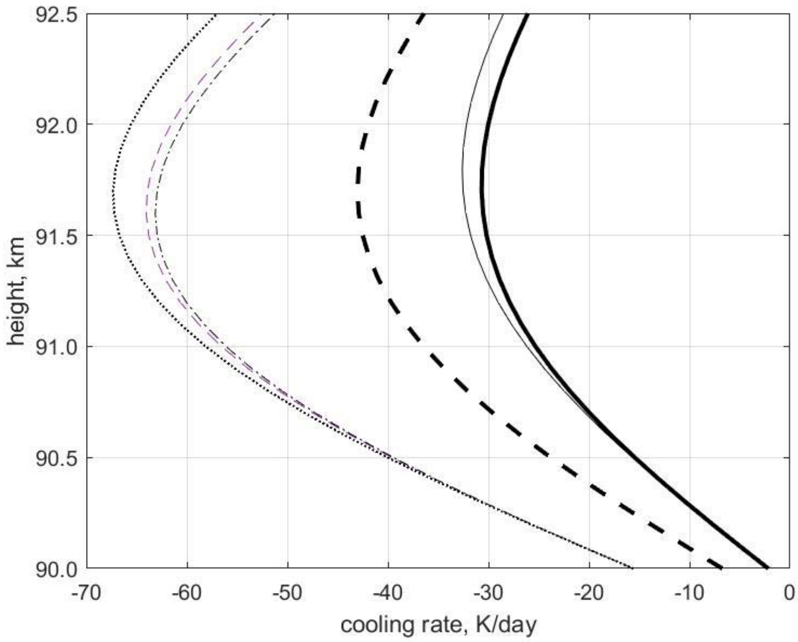

Figure 5. The cooling rates calculated by equation (22) with *G* = 0 K/km – *Ri* = 0.25, *G* = -3K/km – *Ri* = 0.286, *G* = -5 K/km – *Ri* = 0.31, G = -7 K/km – *Ri* = 0.34, *G* = -8 K/km – *Ri* = 0.36, G = -9 K/km – *Ri* = 0.38 (thick solid, dashed and dashed-dotted curves and thin dotted, solid curves and thick dotted curve, respectively) and the Q_{ed} values calculated with *Ri* = 0.25 and the *G* values from -3 K/km to -9 K/km are shown by curves near the thick solid curve.

6. Conclusions

For the first time, by comparing the accelerations in wind shear and the buoyancy force, it is shown that the critical Richardson number, corresponding to the equilibrium of these forces, can

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be estimated and the dynamic instability developed for $Ri < Ri_c$. This new approach is very different from the approach used in classical studies (Miles, 1961) and subsequent papers. Note that Miles and the other authors did not consider the temperature's influence on dynamic instability development. However, the mesosphere is characterized by the negative temperature gradient, and the turbulence peak is observed in this region. For the first time, it has been estimated and established that the Ri_c value depends on the temperature gradient. The Ri_c value increases with the negative mesospheric temperature gradient increase. It should be emphasized that our estimated Ri_c value is exactly the same as the Ri_c value of 0.25 estimated by Miles (1961) and other authors and does not depend on the temperature for $dT/dz = 0$.

The Richardson number dependence on the temperature gradient influences the cooling rates induced by eddy turbulence. These rates significantly increase with an increasing Ri_c, but the influence of the negative temperature gradient on the cooling for $Ri_c = const = 0.25$ is very small.

Also, our results show that criterion Ri_c = 0.25 can be used for turbulent diffusion that is characterized by eddies with a size that is much less than the scale height of the atmosphere. The Ri_c value increases with the increase in the vertical size of the wind shear (see Fig. 3a), but there is a problem with applying the term "eddy diffusion" to momentum and heat transport because of the large-scale eddies in this case (Vlasov and Kelley, 2015).

In general, our results show that the criterion Ri_c = 0.25 can only be applied to turbulence with small scales corresponding to the eddy diffusion. This diffusion provides the mixing of neutral constituents and their diffusive separation as a result of the competition between eddy and molecular diffusion. In this case, the criterion Ri_c = 0.25 is necessary and sufficient, but not for the more complicated shears mentioned above and observed in the lower thermosphere.

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Appendix 1

Derivation of formula (6) in the paper. We start by using the adiabatic equation $pT^{-\gamma/(\gamma-1)} = const$:

$$\frac{\partial}{\partial z}\left[pT^{-\gamma/(\gamma-1)}\right] = 0 \tag{A1}$$

$$p = \rho RT \tag{A2}$$

$$\gamma = Cp/Cv = 1 + 2/N \tag{A3}$$

$$\gamma/(\gamma-1) = 1 + N/2 \tag{A4}$$

$$\frac{\partial}{\partial z}\left[R\rho T \times T^{-1-N/2}\right] = R\left[\frac{\partial \rho}{\partial z}T^{-N/2} - \rho\frac{N}{2}T^{-1-N/2}\frac{\partial T}{\partial z}\right] = 0\,. \tag{A5}$$

Dividing this equation by ρ and multiplying by $T^{-N/2}$, it is possible to get the adiabatic expansion equation

$$\frac{1}{\rho}\frac{\partial \rho}{\partial z} = \frac{N}{2}\frac{1}{T}\frac{\partial T}{\partial z}\,. \tag{A6}$$

Appendix 2

Derivation of formula (17) for $\partial T/\partial z = G = 0$:

$$Ri_c = \frac{\left[1-\frac{g(z-z_0)}{B}\right]^{N/2}}{\left[1-\frac{g(z-z_0)}{B}\right]^{N/2} - exp\left[-\frac{(z-z_0)}{H_A}\right]}\frac{0.5gN(z-z_0)}{B-g(z-z_0)} = \frac{F(z)}{\varphi(z)} \tag{A1}$$

where $B = T_0 C_p(1 + N/2)$ and

$$\frac{\partial F}{\partial z} = -\frac{Ng}{2B}\left[1-\frac{g(z-z_0)}{B}\right]^{N/2-1}\frac{0.5gN(z-z_0)}{B-g(z-z_0)} + \left[1-\frac{g(z-z_0)}{B}\right]^{N/2}\frac{0.5gN[B-g(z-z\_0\ )]+0.5gN(z-z_0)g}{[B-g(z-z_0)]^2}\,. \tag{A2}$$

For $z = z_0$,

$$\frac{\partial F}{\partial z} = \frac{0.5gNB}{B^2} = \frac{0.5gN}{B} \tag{A3}$$

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$$\frac{\partial \emptyset}{\partial z} = -\frac{N_g}{2B}\left[1-\frac{g(z-z_0)}{B}\right]^{N/2-1} + \frac{1}{H_A} exp\left[-\frac{(z-z_0)}{H_A}\right]. \quad \text{(A4)}$$

For $z = z_0$,

$$\frac{\partial \emptyset}{\partial z} = -\frac{Ng}{2B} + \frac{1}{H_A}. \quad \text{(A5)}$$

Finally, we have a very simple formula:

$$Ri = \frac{0.5gN}{B\frac{mg}{\kappa T_0} - 0.5gN} = \frac{0.5N}{\left(1+\frac{N}{2}\right)^2 - 0.5N} = 0.256 \text{ for } N = 5, G = 0 \quad \text{(A6)}$$

and for $G < 0$,

$$\frac{\partial \emptyset}{\partial z} = -\frac{0.5N_g}{B} - \frac{\partial}{\partial z}\left\{\frac{[T_0 - G(z-z_0)]}{T_0}\right\}^{\frac{mg}{\kappa G}-1} = -\frac{0.5N_g}{B} - \left(\frac{mg}{\kappa G} - 1\right)\left(\frac{-G}{T_0}\right) \text{ for } z = z_0 \quad \text{(A7)}$$

$$\frac{\left(\frac{\partial F}{\partial z}\right)}{\left(\frac{\partial \emptyset}{\partial z}\right)} = \frac{0.5gN}{B\left[-\frac{0.5gN}{B} + \frac{mg}{\kappa T_0} - \frac{G}{T_0}\right]} = -\frac{0.5gN}{-0.5gN + g\left(1+\frac{N}{2}\right)^2 - \frac{GB}{T_0}} = \frac{0.5gN}{\left(1+\frac{N}{2}\right)^2 g - 0.5N_g - GC_p(1+N/2)}. \quad \text{(A8)}$$

Appendix 3

The equation used by Hysell et al. (2009, 2012) is

$$N^2 = -\frac{g}{\rho_0}\frac{\partial \rho_0}{\partial z} = \frac{g}{T}\left(\frac{\partial T}{\partial z} + \frac{g}{C_p}\right). \quad \text{(A1)}$$

Here, N^2 is the buoyancy frequency square and ρ_0 is the background density. This equation is incorrect because first, the buoyancy frequency for incompressible fluid is not equal to the frequency for compressible fluid, and second, the background density given by the equation

$$\frac{1}{\rho_0}\frac{\partial \rho_0}{\partial z} = -\frac{1}{T}\left(\frac{\partial T}{\partial z} + \frac{g}{C_p}\right) \quad \text{(A2)}$$

is much larger than the density given by the equation

$$\frac{1}{\rho_A}\frac{\partial \rho_A}{\partial z} = -\frac{1}{T}\left(\frac{\partial T}{\partial z} + \frac{g}{R}\right) \quad \text{(A3)}$$

for hydrostatic equilibrium corresponding to real atmospheric conditions. For example, the scale height of the density is $H = \kappa T(1 + N/2)/mg$ corresponding to equation (A2) where $\partial T/\partial z = 0$ is larger by a factor of 3.5 than the scale height of the background atmospheric density $H = \kappa T/mg$ corresponding to equation (A3). The atmospheric density inferred from equation (A2) with $\partial T/\partial z = G$ is given by the formula

$$\rho_A = \rho_{A0}\{[T_{A0} + G(z - z_0)]/T_{A0}\}^{(-mg/\kappa G(1+0.5N)-1)} . \qquad \text{(A4)}$$

This formula is similar to formula (13b) but with $G > 0$ and $-mg/\kappa G(1 + 0.5N)$ instead of $-mg/\kappa G$. The density given by formula (A4) is much larger than the density given by formula (13b) for $G > 0$. Substituting formula (A4) instead of the exponential term in equation (16) and using L'Hospital's rule, it is possible to get the equation

$$Ri_c = \frac{0.5gN}{g(1+0.5N)-0.5gN+GC_p(1+0.5N)} = \frac{0.5gN}{g+GC_p(1+0.5N)} \qquad \text{(A5)}$$

instead of equation (17).

According to Fig. 2 in Hysell et al. (2012), a sporadic *E* layer with significant irregularities was observed by Arecibo INR at a height of around 110 km at 19:30 – 20:30 LT on July 2, 2010 in the lower thermosphere. The authors used the data on this layer to infer the parameters of the wind shear and then, using a numerical model, they estimated the Ri_c value of 0.75 for the dynamic instability corresponding to the observed irregularities in this region. According to the data shown in Fig. 2 (Hysell et al., 2012), the temperature gradient in the instability at around 110 km is $G =$ 6-8 K/km and the Ri_c value can be found to be 0.8 – 0.65, respectively, according to equation (A5). It follows that the large Ri_c value of 0.75 estimated by the numerical model of Hysell et al. (2012) can only result from the large density used instead of the correct background density. In this case, the Ri_c value does not depend on the specific features of wind shear inferred by the authors and used in the numerical model. According to equation (17) with $G > 0$ and the

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background density given by formula (13b) with $G > 0$, the Ri_c value decreases from 0.25 to 0.2 with G increasing from 0 to 8 K/km.

Competing Interests

The authors declare that they have no conflict of interest.

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