Author Response to the Comments of Reviewer 2:

"Dependence of the critical Richardson number on the temperature gradient in the mesosphere"

Michael N. Vlasov and Michael C. Kelley

Reviewer comment:
Regarding the title, I completely do not get the relationship to the mesosphere, besides the fact that the authors also consider situations with negative vertical gradient of temperature.

Author response:
The object of our study is turbulence in the mesosphere as the region of the upper atmosphere that includes the stratosphere, mesosphere, and thermosphere. The most import feature of the stratosphere and thermosphere is the positive gradient of the temperature. The mesosphere is the only region in the upper atmosphere that is characterized by the temperature negative gradient. The other main features of the mesosphere are the turbulence peak in the upper mesosphere and the wind shear maximum in this region. The negative gradient of the background temperature and the wind shear (Larsen, 2002) (page 5, lines 95-97) in this region provide sufficient and essential conditions for the development of dynamic instability. Turbulence and wind shear are not observed in the thermosphere. According to the observations (for example, Haack et al. (2014)), the very narrow layers of turbulence (localized turbulence) take place in the stratosphere. According to Fig. 9 in Haack et al. (2014), a very large buoyancy frequency and a very small wind shear are observed in these turbulent layers. This indicates the complicated structure of the wind shear (see, for example, Galperin et al. (2007)). Our assumption cannot be applied in this case because of the problem with determining the acceleration of this wind shear.

Revision in the paper:
no changes

Reviewer comment:
Ref#1 also raised this issue and in the AC comment the authors contradict themselves by arguing (on three full pages) that the other studies (like Obukhov (1971)) are not applicable for the mesosphere, but then surprisingly in the final paragraph they write:" ...Ric dependence...is obtained by us without using density, neutral composition, and other parameters of the mesosphere." With some weird remark that the applicability is linked with the uniform turbulence. The connection of the study under review with the mesosphere is demonstrated by the figures, where the x axis shows height about 90 km. But, this is just due to the author arbitrariness connected probably with the choice of temperature values they used for evaluations.

Author response:
Obukhov considers the turbulence in the surface layer. He notes that, “Since the height of the surface layer is not great (on the order of a few tens of meters), the changes of absolute density and temperature within the layer are small and can be considered negligible”. This means neglecting the term $(p_0/p)^{\xi}$ with altitude in the formula $\theta = T(p_0 / p)^{\xi}$ and using the formulas

$$\frac{\partial\theta}{\partial z} = \left(\frac{\partial T}{\partial z} + \frac{g}{C_p}\right) \qquad Ri = \frac{g}{T}\frac{\partial\theta}{\partial z}\left(\frac{\partial v}{\partial z}\right)^{-2}.$$

This approach and these formulas cannot be used for the mesosphere. The thickness of the surface layer considered in the paper is less by a factor of 80 than the scale height of the atmosphere (about 8 km) and this condition is very different from the mesospheric conditions where the scale heights of 4 – 6 km and the thickness of the turbulent layers may be larger than 1 km and the turbulence occupies a region of 40 km. Also, there are other important distinctions between the surface layer in the lower troposphere and the mesosphere. Apparently, the reviewer does not know the principal distinctions between the surface layers in the lower troposphere and the mesosphere.

Revision in the paper:
no changes

Reviewer comment:
Btw. the study of Obukhov (1971) gives a rigorous summary of the Ri and Ric dependence on the temperature gradient and the authors need to explicitly cite this study and show where they give superior scientific information.

Author response:
This reviewer’s statement is wrong. There is only one sentence on estimating the Ri_{cr} value in the paper (page 15): “Corresponding processing of Sverdrup’s data leads to Ri_{cr} = 1/11, which is used later in numerical calculations” and then the author states that, “The determination of the critical *Ri* number is an important problem for atmospheric physics and may be solved only experimentally on the basis of processing reliable data for simultaneous measurements of wind and temperature distributions in the lower layer of the atmosphere”.

Thus, Obukhov uses the experimentally determined value (the only value) of Ri_c for a very rough estimate of the temperature gradient according to his statement (page 21): “Thus, the order of magnitude of the temperature gradient calculated according to K_∞ agrees with the observations. In accordance with Sverdrup’s observations, the value Ri_c = 1/11 was used during calculations of the gradient”. It is necessary to emphasize that no dependence of the Ri_c value on the temperature gradient is presented because the author used the only value of Ri_c = 1/11 that was experimentally determined. This is exactly the opposite of what we have done in our paper. We theoretically define the Ri_c value and calculate the different Ri_c values for the different temperature gradients (see Figs. 3b and 4).

It is necessary to emphasize that Obukhov's result with a huge uncertainty in the temperature gradient calculated for the Ri_c fixed value strongly contradicts the direct and unique dependence of the Ri_c value on the temperature gradient presented in our paper. This contradiction and other problems with estimates using some formulas presented in the paper are explained in the paper by A.S. Monin and A.M. Obukhov, "Turbulent mixing in the atmospheric surface layer" (*Trudy Geophys. Inst*., 1954, N°24, 151 *and "Turbulence and atmospheric dynamics", ed. J.L. Lumley, NASA, CTR Monograph,* November 2001, p. 164). The authors of this paper state that "Obukhov used some insufficiently reliable data (the critical Richardson number was erroneously taken to be 1/11 on the basis of Sverdrup's results) and therefore we could not directly apply his formulas for the practical calculations". This statement is in good agreement with our attempt to use some of the formulas given in Obukhov's paper.

We are very confused by the reviewer's recommendation of this paper, which, according to the author's statement in his next paper, presents the wrong Ri_c value and the wrong formulas are used.

It should be noted that Obukhov's paper was published in 1946 by the journal *Trudy Inst Teor. Geophys,* vol. 1, 95–115. However, this publication was really inaccessible outside of the USSR. The reference given by reviewer 2 corresponds to a translation of this paper published by the journal *Boundary-Layer Meteorol,* 1971, 2, 7-29. In the introduction to this publication, J. A. Businger and A.M. Yaglom explain the reason for this publication: "Probably the major contribution of the paper is the introduction of the 'length scale of the dynamic turbulence sublayer', *L*. This length scale was later introduced independently by Lettau (1949), and at present, it is commonly known as the Monin-Obukhov length. Its fundamental role in the whole field of boundary-layer meteorology was most clearly explained in the well-known paper by Monin and Obukhov (1954)". The authors of the introduction do not mention the problem with the Richardson number in Obukhov's paper because of the comments in Monin and Obukhov (1954) discussed above.

Revision in the paper:
no changes

Reviewer comment:
A) Most importantly, I have serious concern about the validity of the methodology and flawlessness of the analytical derivations in this paper: The crucial point of this study is that the authors assume adiabatic expansion. While this can be a good assumption for the GW induced perturbations, it is completely irrelevant for the background, where e.g., the solar tides govern a significant part of the mesospheric variability. Also, the authors use this assumption to connect the vertical gradient

of full (background + disturbed) density distribution to the full temperature and its gradient and wind shear (Eqs. 6,7,8,9, 10). Also in the light of tides, this assumption crucial for the paper needs to be properly justified, ideally by referencing observational studies.

Author response:
"Turbulence is generated by waves breaking in the MLT through mechanisms such as convective and dynamic instabilities (e.g., Hodges, 1969; Lindzen, 1981; Zhao et al., 2003; Liu et al., 2004; Williams et al., 2006; Hecht et al., 2014)". Hecht, J. H., K. Wan, L. J. Gelinas, D. C. Fritts, R. L. Walterscheid, R. J. Rudy, A. Z. Liu, S. J. Franke, F. A. Vargas, P. D. Pautet, M. J. Taylor, and G. R. Swenson ((2014); The life cycle of instability features measured from the Andes Lidar Observatory over Cerro Pachón on 24 March 2012, *J. Geophys. Res. Atmos*., 119, 8872–8898, doi:10.1002/2014JD021726". (Guo, Y., A. Z. Liu, and C. S. Gardner (2017), *Geophys. Res. Lett*., 44, 5782–5790,doi:10.1002/2017GL073807.)

Hodges (*J. Geophys. Res*., 72, 3455-3458, 1967) pointed out that it is unlikely to have conditions for dynamic instability without gravity waves. Tides alone are not sufficient to induce dynamic or convective instabilities, but the tides can influence the conditions for dissipation of the gravity waves and the development of dynamic instability due to change in the temperature gradient. In any case, adiabatic expansion is a fundamental process for dynamic instability and the adiabatic lapse rate is a very important parameter. This assumption is used to derive the buoyancy frequency formula (see, for example, Peixoto, J. P., and Oort, A. H.: *Physics of Climate.* New York: Springer-Verlag, 1992), which is included in the chain of equations (6)–(10). The Richardson number depends directly on the adiabatic lapse. Unfortunately, the reviewer does not explain why adiabatic expansion cannot exist for the tides. We do not consider the mesospheric background parameters' variability induced by the different processes. We only consider the dependence of dynamic instability on the temperature gradients in the mesosphere. Unfortunately, the reviewer does not explain what kind of observational studies he means. In our paper, the results of the experimental data (Bishop et al., 2004; Kelley et al., 2003; Larsen, 2002; Lubken, 1997) are used.

Revision in the paper:
no changes

Reviewer comment:
But more than just general doubts about the validity of this assumption, the authors make errors also in analytical description, where in eq. 8, which shows partial derivative of T with altitude they refer to it as (P4L81) "temperature gradient in the parcel (sic) with upward motion and adiabatic expansion" - but for this, total derivative would have to be shown.

Author response:
We are very surprised by this comment. Eq. 8 is the result of the simple combination of generally accepted Eqs. 2, 6, and 7 with partial derivatives and it is impossible to obtain this formula with total derivatives in only one equation in this combination. Eq. 6 is the key formula and presents the temperature gradient corresponding to adiabatic expansion due to upward parcel displacement. This result does not depend on the kinetics of parcel motion. This is the generally accepted approach for estimating the effect of parcel displacement on the temperature for adiabatic expansion/compression. Unfortunately, the reviewer's statement is too general without an explanation or a reference.

Revision in the paper:
no changes

Reviewer comment:
Most importantly, on their way from eq. 6 to 10 they use in P4L80 an equation for Ri based on different assumption (they don't tell anything about this formula, which is crucial) and then they consider this Ri (general?) to be equal to the Ri in eq. 7 (adiabatic expansion) for deriving eq. (10).

Author response:
The derivation of Eq. 6 was given in Appendix 1. Taking this comment into account, an additional explanation is included in the text (page 3) and Appendix 1. The main point is that Eq. 4 corresponds to incompressible fluid and $\omega_B^2=(-g/\rho_0)\partial\rho_0/\partial z$, but Eq. 6 corresponds to compressible fluid (adiabatic expansion) and $\omega_B^2 = (g/T)(\partial T/\partial z + g/Cp)$ should be used, so in this case, Eq. 7 and Eq. 8 must correspond to compressible fluid.

Revision in the paper:
page 3, lines – 57, 59, 63
page 4, lines 72, 73, 80, 81, 83, 90
page 16, lines – 262, 263, 264, 266

Reviewer comment:
A similar situation takes place in section 3, where they give equation 13b (P6L110) without properly discussing how they derived this equation and the underlying assumptions (polytropic atmosphere?). This formula (13b) and the formula for wind shear (eq. 10) are the crucial parts of the paper, because every other result then presented is only a trivial evaluation of Ri based on those formulas.

Author response:
We did not show the derivation of formula (13b) because this formula is the same as the well-known and commonly used formula (Banks and Kockarts, 1973, part A, page 36, 1973):

$$\rho = \rho_0 \left(H / H_o \right)^{-(1+\beta)/\beta} \quad \text{(A1)}$$

where $H = \kappa T/mg$, $\alpha = \beta = \partial H/\partial z = (\kappa/mg)\partial T/\partial z$, *and* $n = \rho/m$.

The derivation of eq. 13b is now given in Appendix 4.

Revision in the paper:
Page 19, lines 322-340.

Reviewer comment:
The authors need to carefully rewrite all of their analytical derivations, distinguish properly between local and total derivatives, list the assumptions made an ensure consistency between the assumptions and also distinguish in their formulas between constants and functions of altitude (f(z)). Without this it doesn't make sense to discuss any results given later in the text (poor evaluation of the derived formulas), because my personal opinion (the authors are welcomed to prove otherwise) is that the results are dominated by flaws in their analytical construct.

Author comment:
The reviewer's negative comments are too general without any evidence, examples, or references. For instance, the reviewer says that "the results are dominated by flaws" but does not prove his/her mere allegations. Moreover, the reviewer has stated (in two separate instances) that the assumptions have not been explicitly listed in the paper, whereas in fact, they were provided on pages P2L27,28; P3L58-64; P4L72,73; P5L96,97; P6L 111-113; P7L114,115 and L126,127; and P13L204-206 of the submitted manuscript. Also, it is totally unclear why the reviewer insists on using "total derivatives" while all the well-known formulas are customarily defined in terms of partial derivatives.

Revision in the paper:
no changes

Reviewer comment:
Language: Non-scientific language is used frequently, with weird phrases like: we could find just one paper… or acceleration in wind shear the authors write that some study is wrong, but do not prove it. Just to list: What is the? P5L92 Does wind shear really induce vertical accelerations? (no, you have to replace the word induce by e.g., support) Page 3, L 67 not wind shear nor stability

are forces. Those were the most striking ones. I am not listing all the typos made in the manuscript because I expect major changes before it can be assessed for publication.

Author response:
Note that reviewer 1 did not have a problem with the language used in our paper. We made a few language corrections to the text. The reviewer's statement, "the authors write that some study is wrong, but do not prove it," is incorrect. The explanation was presented in detail in Appendix 3. Note that this reviewer's statement does not demonstrate a language problem. Our paper stated, "The goal of this paper is to estimate the critical Richardson number, Ri_c, corresponding to the equilibrium between the buoyancy force and the force induced by wind shear in the mesosphere. Dynamic instability is developed for $Ri < Ri_c$. Our approach considers the acceleration corresponding to both forces, taking into account the mesospheric temperature height distributions". It is not clear why the reviewer objects to the word "force". Again, note that reviewer 1 did not have a problem with the language used in our paper.

In general, reviewer 2's apparent lack of understanding concerning the distinction between the surface layer in the troposphere and the mesosphere, the unproven statements about the important role of tides for dynamic instability development, the use of total derivatives in commonly used formulas, and his/her request to present the derivation of the well-known and commonly used formula of density distribution in the upper atmosphere clearly demonstrate that the reviewer is not adequately familiar with the physics of the upper atmosphere and dynamic instability. One obvious evidence of this is the reviewer's persistent recommendation of a paper that, according to the author's statement in his next paper, presents the wrong Ri_c value and uses the wrong formulas.

Revision in the paper:
Changes were made, including on page 3.

Dependence of the critical Richardson number on the temperature gradient in the mesosphere

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Abstract

Maximum upper atmospheric turbulence results in the mesosphere from convective and/or dynamic instabilities induced by gravity waves. For the first time, by comparing the vertical accelerations induced by wind shear and the buoyancy force, it is shown that the critical Richardson number Ri_c can be estimated. Dynamic instability is developed for $Ri < Ri_c$. This new approach, for the first time, makes it is possible to establish and estimate the temperature gradient impact on dynamic instability development. Regarding our results, Ri_c increases from 0.25 to 0.38 as the negative temperature vertical gradient increases from $\partial T/\partial z = 0$ to $\partial T/\partial z \leq -9$ K/km. However, Ri_c for the temperature, independent of altitude, is 0.25, coinciding exactly with the Ri_c commonly used and estimated in classical studies (Miles, 1961; Howard, 1961) and subsequent papers without the temperature impact. The increase in the Ri_c value strongly influences cooling, inducing the cooling rate increase. Also, our results show that criterion $Ri_c < 0.25$ can only be used for the turbulent diffusion, which is characterized by eddies with sizes much smaller than the scale height of the atmosphere. The Ri_c value increases with the increasing size of the eddies, but the term "eddy diffusion" cannot be applied to transport due to the large-scale eddies (Vlasov and Kelley, 2015).

Key Words: Richardson number, dynamic instability, turbulent cooling, mesosphere

1. Introduction

In general, the Richardson number Ri can be defined as the ratio of the destruction of turbulent kinetic energy by buoyancy forces and the production of turbulent energy by the wind shear flow. This determination leads to the relation

$$Ri = \omega_B^2/S^2 \,, \tag{1}$$

where ω_B is the buoyancy frequency,

$$\omega_B^2 = \frac{g}{T}\left(\frac{\partial T}{\partial z} + g/C_p\right) , \tag{2}$$

and T is the temperature, g is the acceleration of gravity, C_p is the heat capacity at constant pressure, and

$$S = \frac{\partial V}{\partial z} \tag{3}$$

is the vertical shear of the horizontal wind with the velocity $V(z)$ height profile. It is generally accepted that a dynamic instability develops when the Richardson number is less than ¼, i.e., the parcel's vertical motion induced by wind shear dominates the motion induced by the buoyancy force. The former creates and the latter destroys these perturbations. Most authors use the critical Richardson number Ri_c < ¼ without references. Some authors refer to Miles (1961) and Howard (1961). They consider the stable-stratified, horizontal shear flows of an ideal fluid. A set of studies takes into account the time-dependent shear flow and the results of laboratory experiments (Peixoto and Oort, 1992; Galperin et al., 2007). However, we could not find papers on the critical Richardson number that take the mesospheric conditions into account. Miles and other authors (Abarbanel et al., 1984; Ligniéres et al., 1999; Galperin et al., 2007) did not consider the temperature's influence on the Ri_c value. However, the eddy turbulence peak is observed in the mesosphere or the lower thermosphere where the large negative and positive gradients of the temperature occur. We could find just one paper [Hysell et al., 2012] on the estimate of the Ri_c value in the lower thermosphere. Using the data on observations of the sporadic *E* layer, Hysell et al. (2012) inferred the parameters of wind shear corresponding to the irregularities observed in the layer and estimated the Ri_c value of 0.75. However, the authors used the wrong formula for the background density, resulting in densities much larger than the observed atmospheric density corresponding to the hydrostatic equilibrium. It is shown in Appendix 3 how $0.7 < Ri_c < 0.8$ can be found due to the background density used by Hysell et al. (2012).

The principal measure of stability regarding the buoyancy effects of the density gradient for overriding its inertial effects **in the incompressible fluid** is the Richardson number given by formula (1) in Miles (1961), which can be written as

$$Ri = -g\frac{\partial \rho_0}{\partial z} / \left\{ \rho_0 \left[\frac{\partial V}{\partial z} \right]^2 \right\} \tag{4}$$

where ρ_0 is the density and V is the horizontal wind velocity. This formula can be rewritten as

$$\left(\frac{\partial V}{\partial z}\right)^2 = -\frac{g}{Ri}\frac{1}{\rho}\frac{\partial \rho}{\partial z} \ . \tag{5}$$

This will be used here to estimate the accelerations due to wind shear and the buoyancy force **in compressible fluid** under mesospheric conditions.

The goal of this paper is to estimate the critical Richardson number, Ri_c, corresponding to the equilibrium between the buoyant force and the force **supported** by wind shear in the mesosphere.

Dynamic instability is developed for $Ri < Ri_c$. Our approach considers the acceleration corresponding to both forces, taking into account the mesospheric temperature height distributions.

2. Acceleration Induced by Wind Shear

We start from formula (5) **corresponding to compressible fluid, and adiabatic expansion should be taken into account in the mesosphere**. Differentiating the adiabatic relation $pT^{-\gamma/(\gamma-1)} = const$ corresponding to Poisson's equation where $p = \rho\kappa T/m$ and p is the pressure; m is the mean molecular mass; $\gamma = C_p/C_v$; C_p and C_v are the heat capacities at constant pressure and volume; $\gamma/(\gamma-1) = 1 + N/2$; $N = 5$ is the number of degrees of freedom for diatomic gas; and κ is the Boltzmann's constant, it is possible to get the adiabatic expansion equation

$$\frac{1}{\rho}\frac{\partial\rho}{\partial z} = \frac{N}{2}\frac{1}{T}\frac{\partial T}{\partial z} \tag{6}$$

(see the derivation of this formula in Appendix 1). **It is necessary to note that formula (6) corresponds to compressible fluid and, according to** (5):

$$\left(\frac{\partial V}{\partial z}\right)^2 = -\frac{g}{Ri}\frac{N}{2T}\frac{\partial T}{\partial z}\ . \tag{7}$$

This formula corresponds to compression fluid. Taking into account $Ri(\partial V/\partial z)^2 = \omega_B^2 = (g/T)\left(\partial T/\partial z + g/C_p\right)$ and using formula (6), the temperature gradient in the parcel with upward motion and adiabatic expansion can be given by the equation

$$\frac{\partial T}{\partial z} = -\frac{g}{(1+N/2)C_p} \tag{8}$$

and

$$T = T_0 - \frac{g}{\left(1+\frac{N}{2}\right)C_p}(z - z_0)\ . \tag{9}$$

Note that $\omega_B{}^2$ = $(-g/\rho_0)\partial\rho_0/\partial z$ corresponds to incompressible fluid (see Appendix 3 for details). By substituting formulas (8) and (9) in formula (7) multiplied by $(z - z_0)$, it is possible to obtain the formula

$$a_{ws} = \frac{g^2 N(z-z_0)}{2Ri\left[T_0 C_p(1+N/2) - g(z-z_0)\right]} \tag{10}$$

where

$$a_{ws} = \left(\frac{\partial V}{\partial z}\right)^2 (z - z_0) \tag{11}$$

is the acceleration in wind shear. As can be seen from Fig. 1, this acceleration increases with the increase of the vertical size of the wind shear layer. Note that this size cannot exceed 1–2 km according to the experimental data (Larsen, 2002). The a_{ws} dependence on the altitude is linear because $g(z - z_0) \ll T_0 C_p (1 + N/2)$ for $-z_0 < 2$ km.

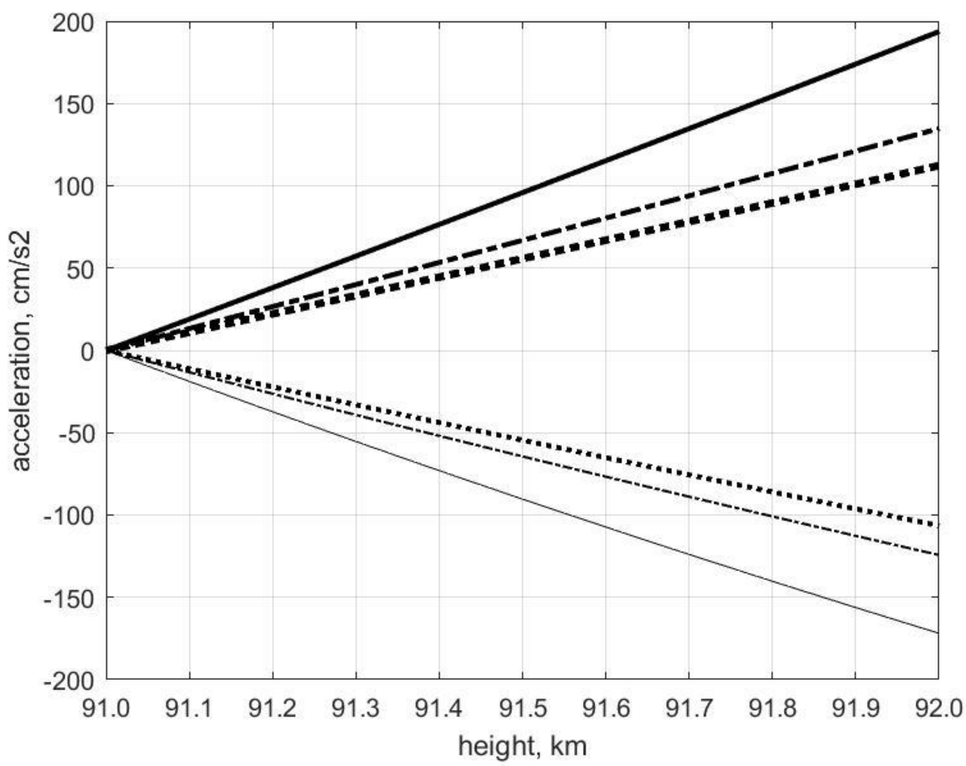

Figure 1. The height profiles of the wind shear $a_{ws} > 0$ and buoyant $a_B < 0$ accelerations calculated by formulas (11) and (15), respectively, with T_0 = 140 K and Ri_c = 0.25 (solid curves), with T_0 = 140 K and $Ri_c = R/C_p$= 0.286 (dashed-dotted curves), and with T_0 = 200 K and Ri_c = 0.286 (dotted curves).

3. Acceleration Induced by the Buoyancy Force

The buoyancy force is $F_B = g(\rho_A - \rho_D)$ where ρ_A and ρ_D are the background atmospheric density and the disturbed density, respectively. The acceleration is given by

$$a_B = g[(\rho_A - \rho_D)\rho_D] \,. \tag{12}$$

The atmospheric density distribution can be given by

$$\rho_A = \rho_{A0} exp[-(z - z_0)/H_A] \tag{13a}$$

for $dT_A/dz = 0$ in the mesopause and the formula

$$\rho_A = \rho_{A0}\{[T_{A0} - G(z - z_0\,)\,]/T_{A0}\}^{(mg/\kappa G - 1)} \tag{13b}$$

for $dT_A/dz = G < 0$ below the mesopause, and $H_A = \kappa T_{A0}/\mathrm{mg}$ is the scale height of the atmospheric gas. By integrating equation (6) with the temperature and temperature gradient given by formulas (8) and (9)**,** it is possible to get the disturbed density distribution ($T_0 = T_{A0}$),

$$\rho_D = \rho_{A0}\left[\frac{T_0 - \frac{G(z-z_0\,)}{Cp(1+N/2)}}{T_0}\right]^{N/2} \quad , \tag{14}$$

and the acceleration corresponding to the buoyancy force can be written as

$$a_{\boldsymbol{B}} = g\left[\left(\frac{\rho_A}{\rho_D}\right) - 1\right] = g\,\frac{\rho_{A0} e^{-\frac{(z-z_0)}{H_A}}}{\rho_{A0}\left[\frac{T_0 - \frac{g(z-z_0\,)}{Cp\left(1+\frac{N}{2}\right)}}{T_0}\right]^{\frac{N}{2}}} - g \tag{15}$$

for $dT_A/dz = 0$. As seen from Fig. 1, there is very good agreement between the a_{ws} and a_B absolute values for Ri_c = 0.25, and T_0 = 140 K and T_0 = 200 K for the vertical size of a stable wind shear layer that is less than 400 m. The a_{ws} value becomes larger than the a_B value for $z - z_0 >$ 400 m, which means that the Ri_c value should be increased. The turbulence develops if α_{ws} is larger than the α_B that corresponds to $Ri < Ri_c$. We emphasize that the perturbation scale sizes induced by wind shear do not exceed 1-2 km, according to the observations (see Lübken (1997)). Note that formula (13b) should be used instead of formula (13a) in the nominator of formula (15) for atmospheric temperature distribution with $\frac{dT_A}{dz} < 0$. As can be seen from Fig. 2, the a_B values significantly decrease in this case, since the atmospheric density given by formula (13b) is larger and the density gradient is less than the density and gradient corresponding to formula (13a). The small buoyancy force corresponds to the small density gradient. This dependence explains the a_B reduction with the T_A decrease.

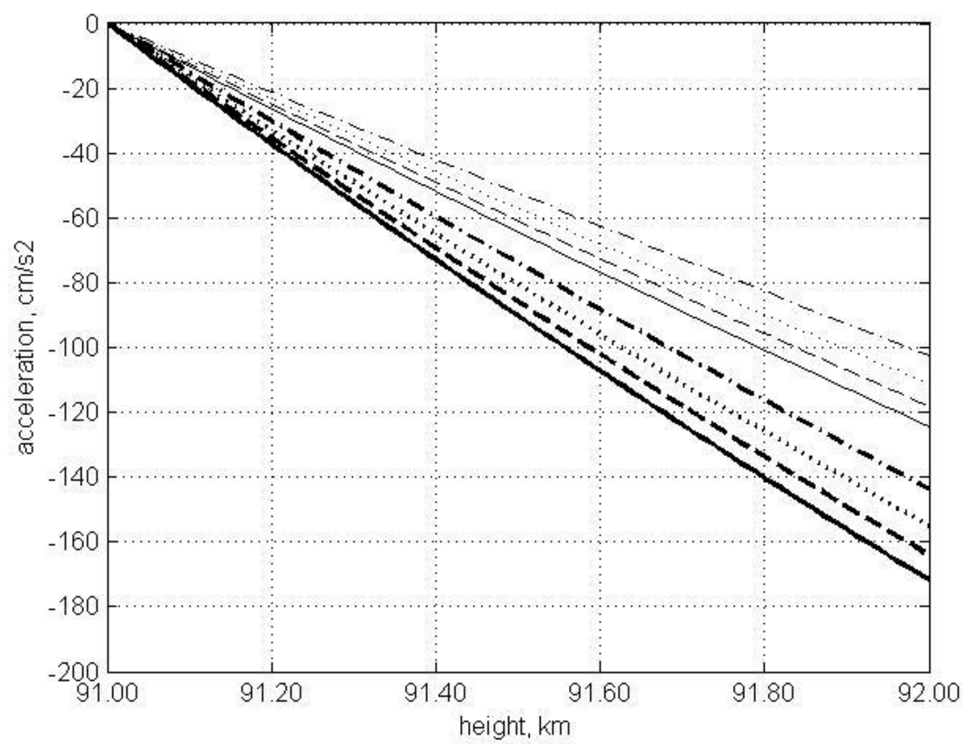

Figure 2. The height profiles of the acceleration of the buoyancy force calculated by formula (15) with the nominator $\rho_{A0}\{[T_{A0} - G(z - z_0)]/T_{A0}\}^{(mg/\kappa G - 1)}$ for $T_0 = T_{A0}$ = 140 K and 200 K (thick and thin curves, respectively) and G = 1, 2.8, and 5 K/km (dashed, dotted and dashed-dotted curves, respectively), and calculated by formula (15) (solid curves).

4. Estimating the Richardson Number

Using formulas (11) and (15) in the equation $a_{ws} + a_B = 0$, the formula for Ri_c can be inferred:

$$Ri_c = \frac{\left[1 - \frac{g(z-z_0)}{T_0 C_p(1+N/2)}\right]^{N/2} \frac{gN(z-z_0)}{2C_p(1+N/2)\left[T_0 - \frac{g(z-z_0)}{C_p(1+N/2)}\right]}}{\left[1 - \frac{g(z-z_0)}{T_0 C_p(1+N/2)}\right]^{N/2} - exp\left[-\frac{(z-z_0)}{H_A}\right]} \ . \qquad (16)$$

The Ri_c values calculated by formula (16) and this formula with $\{[T_0 - G(z - z_0)]/T_0\}^{(mg/\kappa G-1)}$ (see formula (13b) instead of the exponential term) are shown in Figs. 3a and 3b. The Ri_c values increase with increasing altitude, corresponding to the vertical expansion of the region of the stable wind shear. However, according to the experimental data (Larsen, 2002; Kelley et al., 2003; Bishop et al., 2004), the wind shears are very unstable. As mentioned above, the size scales of the density perturbations do not exceed 1 – 2 km, according to the observations. A more accurate consideration of eddy turbulence (Vlasov and Kelley, 2015) concludes that the scale size of density perturbations l should be much less than the scale height of atmospheric gas, $l << H_A$ and $l << 4$ km for $T_A = T_0 = 140$ K and $l << 5.7$ km for $T_A = T_0 = 200$ K. However, this restriction can only apply to turbulence corresponding to the eddy diffusion approximation (Vlasov and Kelley, 2015). As seen from Fig. 3a, the Ri_c value of 0.25 corresponds to perturbations with scales less than 10 m, and the Ri_c values reach 0.256 and 0.263 for l = 200 m and 400 m and for $T_{A0} = 140$ K and 0.254 and 0.257 for $T_0 =$ 200 K, respectively. The Ri_c value of 0.25 corresponds to the mean value l = 27.3 m obtained by Lübkin (1997), using the measured spectrum of the density fluctuation. Vlasov and Kelley (2015) reconsidered the results of Kelley et al. (2003) and found that the spectrum scale fluctuations inferred from the meteor train turbulence observations can be approximated by Heisenberg's formula with l = 119 m, and eddies with very large scales may occur in the narrow layer of localized turbulence. As can be seen from Fig. 3b, the Ri_c values increase with the increase in the negative gradient of the temperature and can reach almost 0.36.

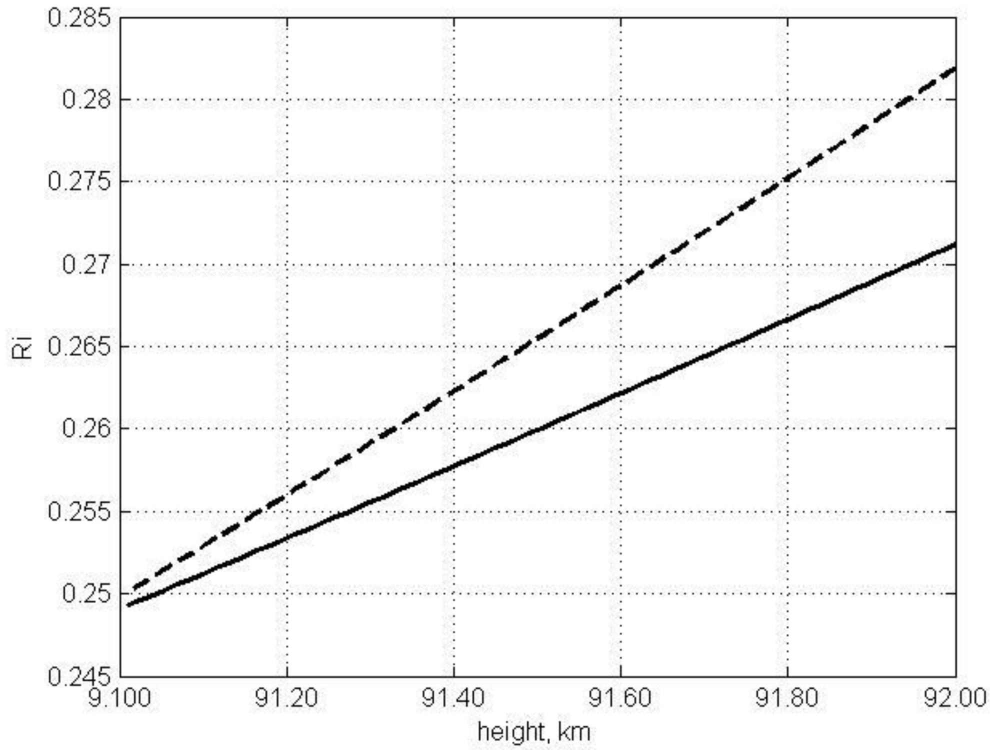

Figure 3a. The height profiles of the critical Richardson number calculated by formula (16) with T_0 = 140 K and 200 K (dashed and solid lines, respectively).

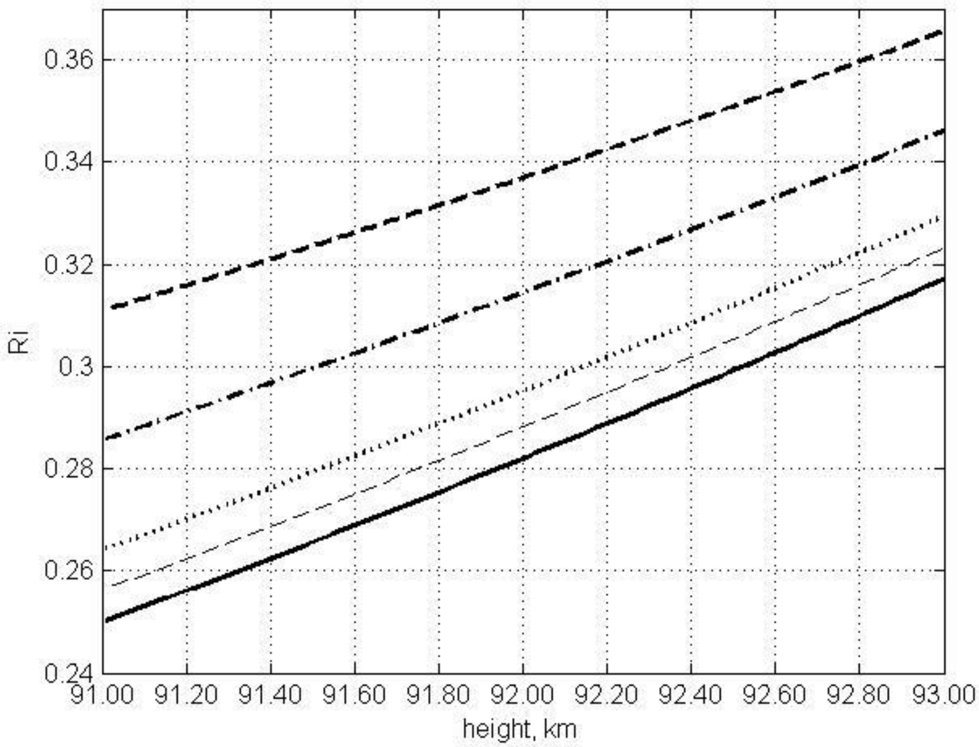

Figure 3b. The height profiles of the critical Richardson number calculated by formula (16) with $\{[\boldsymbol{T_0 - G(z - z_0)}]/\boldsymbol{T_0}\}^{(\boldsymbol{mg/\kappa G - 1})}$ instead of the exponential term for the T_0 = 140 K with $dT/dz = G < 0$ with $|G|$ = 0.2, 1, 3, and 5 K/km (dashed thin, dotted, dashed-dotted and dashed thick curves, respectively) and calculated by formula (16) (solid thick curve).

Thus, turbulence can develop with $Ri_c > 0.25$ for wind shears with a vertical size of 1–2 km, but this turbulence may not correspond to eddy diffusion. The scales of the density fluctuations are very small (for example, see Lübken (1997)) that correspond to $z \rightarrow z_0$. However, the Ri_c value estimation for $z \rightarrow z_0$ is problematic because, in this case, the numerator and denominator in formula (16) try to attain zero. This uncertainty can be solved using L'Hospital's rule, leading to the formula (see Appendix 2)

$$Ri_c = \frac{0.5gN}{g(1+N/2)^2 - 0.5gN - GC_p(1+N/2)} \quad (17)$$

for the $\boldsymbol{Ri_c}$ limit value for $\boldsymbol{z \to z_0}$. This formula corresponds to the limit value formula (16) with the term $\{[\boldsymbol{T_0 - G(z - z_0)}]/\boldsymbol{T_0}\}^{(\boldsymbol{mg/\kappa G - 1})}$ instead of the term $exp[-(z - z_0)/H_A]$. The Ri_c dependence on the negative temperature gradient, given by formula (17), is shown in Fig. 4. The *G* increase improves the conditions for the dynamic instability development. Note that the $\boldsymbol{Ri_c}$ value for *G* = 0 coincides with the results of Miles (1961) and the commonly used value of $\boldsymbol{Ri_c}$.

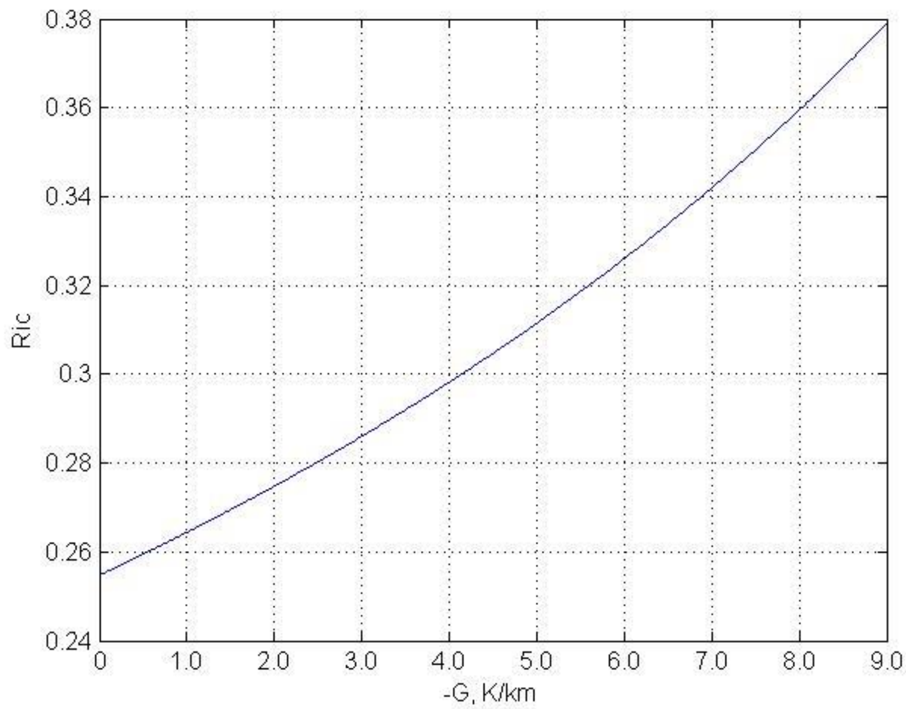

Figure 4. The dependence of the Richardson number $\boldsymbol{Ri_c}$ on the temperature negative gradient calculated by formula (17).

5. The Influence of $\boldsymbol{Ri_c}$ Dependence on *G* on Cooling in the Mesosphere

The eddy turbulence heating/cooling rate can be given by the equation (Vlasov and Kelley, 2010)

$$Q_{ed} = \frac{\partial}{\partial z}\left[K_{eh} C_p \rho \left(\frac{\partial T}{\partial z} + \frac{g}{C_p}\right)\right] + K_{eh}\rho \frac{g}{Tb}\left(\frac{\partial T}{\partial z} + \frac{g}{C_p}\right), \tag{18}$$

where K_{eh} is the coefficient of the eddy heat transport, ρ is the undisturbed gas density, and b is a dimensionless constant given by the relation obtained using the results of Gordiets et al. (1982),

$$b = Ri_c/(P - Ri_c) \tag{19}$$

where P is the turbulent Prandtl number. According to equation (18), the Q_{ed} value is given in units $erg \times cm^{-3} \times s^{-1}$. The K_{eh} value is given by

$$K_{eh} = b\varepsilon/\omega_B^2, \tag{20}$$

where ε is the energy dissipation rate, and b can be given by formula (19). The vertical distribution of the ε value in the turbulent layer can be approximated by the Gaussian function

$$\varepsilon = \varepsilon_m exp[-(z - z_m)^2/h^2], \tag{21}$$

where h is half of the layer thickness and ε_m is the ε value at the altitude of the layer peak z_m. Using this approximation, dividing equation (18) by ρC_p and substituting formula (20) with $b = Ri/(P - Ri)$ and $T = T_0 + G(z - z_0)$, equation (18) can be written in units K/s as

$$Q_{ed} = \varepsilon_m exp\left[-\frac{(z-z_m)^2}{h^2}\right]\left\{\frac{[T_0 + G(z-z_0)]}{g\left(\frac{P}{Ri_c}-1\right)}\left[-\frac{2(z-z_m)}{h^2} - \frac{\frac{mg}{\kappa}}{T_0+G(z-z_0)}\right] + \frac{1}{C_p}\right\}. \tag{22}$$

Using the Ri_c dependence on the temperature gradient given by formula (17), the impact of the Richardson number on the cooling rates can be estimated. According to the results in Fig. 5, the cooling rates increase by a factor of 2.2 for $0.25 < Ri_c < 0.38$ corresponding to $0 \leq G \leq -9$ K/km, but the G value influence on the cooling for Ri_c = const = 0.25 is very small (curves near the thick solid curve). Note that the turbulence induced by the large wind shear may not correspond to the eddy diffusion heat transport. The values of ε_m, z_m, and h correspond to the experimental data (Lübken, 1997).

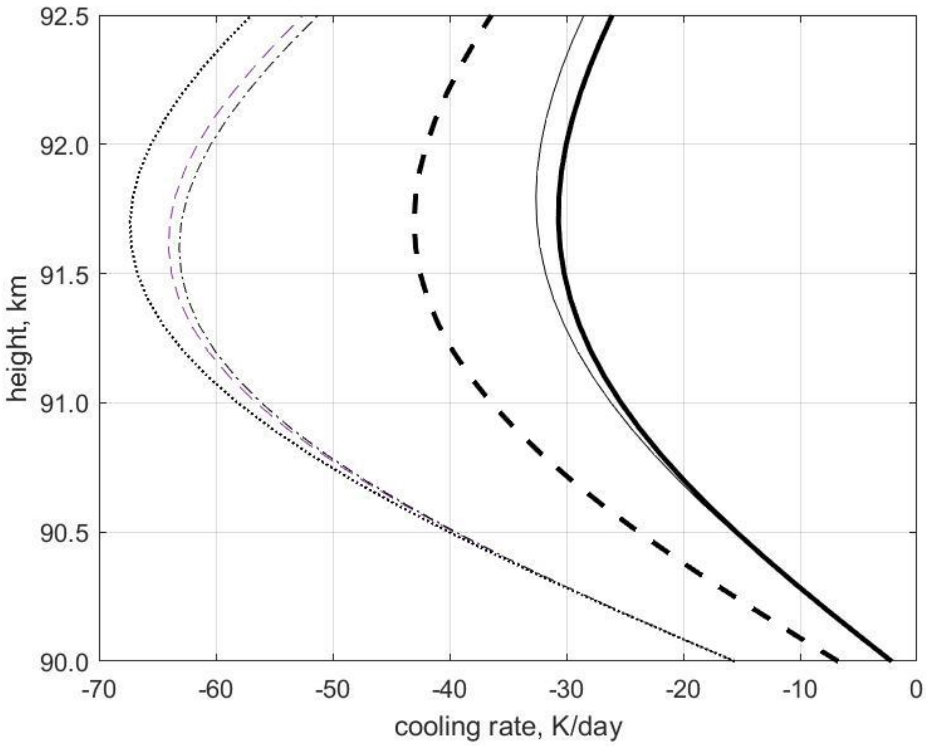

Figure 5. The cooling rates calculated by equation (22) with G = 0 K/km – Ri = 0.25, G = -3K/km – Ri = 0.286, G = -5 K/km – Ri = 0.31, G = -7 K/km – Ri = 0.34, G = -8 K/km – Ri = 0.36, G = -9 K/km – Ri = 0.38 (thick solid, dashed and dashed-dotted curves and thin dotted, solid curves and thick dotted curve, respectively) and the Q_{ed} values calculated with Ri = 0.25 and the G values from -3 K/km to -9 K/km are shown by curves near the thick solid curve.

6. Conclusions

For the first time, by comparing the accelerations in wind shear and the buoyancy force, it is shown that the critical Richardson number, corresponding to the equilibrium of these forces, can be estimated and the dynamic instability developed for $Ri < Ri_c$. This new approach is very different from the approach used in classical studies (Miles, 1961) and subsequent papers. Note that Miles and the other authors did not consider the temperature's influence on dynamic instability development. However, the mesosphere is characterized by the negative temperature gradient, and the turbulence peak is observed in this region. For the first time, it has been estimated and established that the Ri_c value depends on the temperature gradient. The Ri_c value increases with the negative mesospheric temperature gradient increase. It should be emphasized that our estimated Ri_c value is exactly the same as the Ri_c value of 0.25 estimated by Miles (1961) and other authors and does not depend on the temperature for $dT/dz = 0$.

The Richardson number dependence on the temperature gradient influences the cooling rates induced by eddy turbulence. These rates significantly increase with an increasing Ri_c, but the influence of the negative temperature gradient on the cooling for $Ri_c = const = 0.25$ is very small.

Also, our results show that criterion Ri_c = 0.25 can be used for turbulent diffusion that is characterized by eddies with a size that is much less than the scale height of the atmosphere. The Ri_c value increases with the increase in the vertical size of the wind shear (see Fig. 3a), but there is a problem with applying the term "eddy diffusion" to momentum and heat transport because of the large-scale eddies in this case (Vlasov and Kelley, 2015).

In general, our results show that the criterion Ri_c = 0.25 can only be applied to turbulence with small scales corresponding to the eddy diffusion. This diffusion provides the mixing of neutral constituents and their diffusive separation as a result of the competition between eddy and molecular diffusion. In this case, the criterion Ri_c = 0.25 is necessary and sufficient, but not for the more complicated shears mentioned above and observed in the lower thermosphere.

Appendix 1

Derivation of formula (6) in the paper. We start by using the adiabatic equation $pT^{-\gamma/(\gamma-1)} = const$:

$$\frac{\partial}{\partial z}\left[pT^{-\gamma/(\gamma-1)}\right] = 0 \qquad \text{(A1)}$$

$$p = \rho R T \qquad \text{(A2)}$$

$$\gamma = Cp/Cv = 1 + 2/N \qquad \text{(A3)}$$

$$\gamma/(\gamma-1) = 1 + N/2 \qquad \text{(A4)}$$

$$\frac{\partial}{\partial z}\left[R\rho T \times T^{-1-N/2}\right] = R\left[\frac{\partial \rho}{\partial z}T^{-N/2} - \rho\frac{N}{2}T^{-1-N/2}\frac{\partial T}{\partial z}\right] = 0\ . \qquad \text{(A5)}$$

Dividing this equation by ρ and multiplying by $T^{-N/2}$, it is possible to obtain the adiabatic expansion equation

$$\frac{1}{\rho}\frac{\partial \rho}{\partial z} = \frac{N}{2}\frac{1}{T}\frac{\partial T}{\partial z}\ . \qquad \text{(A6)}$$

Using formula (5) in the text and combining formula (2) in the text corresponding to the compressible fluid with equation (6), it is possible to obtain the equation

$$\frac{N}{2}\frac{1}{T}\frac{\partial T}{\partial z} = -\frac{1}{T}\frac{\partial T}{\partial z} - \frac{g}{TC_p} \qquad \textbf{(7)}$$

and the temperature gradient in the parcel with adiabatic expansion can be found to be

$$\boldsymbol{\frac{\partial T}{\partial z} = -\frac{g}{(1+N/2)C_p}}\ . \qquad \text{(8)}$$

Appendix 2

Derivation of formula (17) for $\partial T/\partial z = G = 0$:

$$Ri_c = \frac{\left[1-\frac{g(z-z_0)}{B}\right]^{N/2}}{\left[1-\frac{g(z-z_0)}{B}\right]^{N/2} - exp\left[-\frac{(z-z_0)}{H_A}\right]} \frac{0.5gN(z-z_0)}{B-g(z-z_0)} = \frac{F(z)}{\varphi(z)} \quad \text{(A1)}$$

where $B = T_0 C_p(1 + N/2)$ and

$$\frac{\partial F}{\partial z} = -\frac{Ng}{2B}\left[1-\frac{g(z-z_0)}{B}\right]^{N/2-1} \frac{0.5gN(z-z_0)}{B-g(z-z_0)} + \left[1-\frac{g(z-z_0)}{B}\right]^{N/2} \frac{0.5gN[B-g(z-z\_0\ )]+0.5gN(z-z_0)g}{[B-g(z-z_0)]^2}\,.$$

(A2)

For $z = z_0$,

$$\frac{\partial F}{\partial z} = \frac{0.5gNB}{B^2} = \frac{0.5gN}{B} \quad \text{(A3)}$$

$$\frac{\partial \emptyset}{\partial z} = -\frac{N_g}{2B}\left[1-\frac{g(z-z_0)}{B}\right]^{N/2-1} + \frac{1}{H_A} exp\left[-\frac{(z-z_0)}{H_A}\right]. \quad \text{(A4)}$$

For $z = z_0$,

$$\frac{\partial \emptyset}{\partial z} = -\frac{Ng}{2B} + \frac{1}{H_A}\,. \quad \text{(A5)}$$

Finally, we have a very simple formula:

$$Ri = \frac{0.5gN}{B\frac{mg}{\kappa T_0} - 0.5gN} = \frac{0.5N}{\left(1+\frac{N}{2}\right)^2 - 0.5N} = 0.256 \text{ for } N = 5, G = 0 \quad \text{(A6)}$$

and for $G < 0$,

$$\frac{\partial \emptyset}{\partial z} = -\frac{0.5N_g}{B} - \frac{\partial}{\partial z}\left\{\frac{[T_0 - G(z-z_0)]}{T_0}\right\}^{\frac{mg}{\kappa G}-1} = -\frac{0.5N_g}{B} - \left(\frac{mg}{\kappa G} - 1\right)\left(\frac{-G}{T_0}\right) \text{ for } z = z_0 \quad \text{(A7)}$$

$$\frac{\left(\frac{\partial F}{\partial z}\right)}{\left(\frac{\partial \emptyset}{\partial z}\right)} = \frac{0.5gN}{B\left[-\frac{0.5gN}{B} + \frac{mg}{\kappa T_0} - \frac{G}{T_0}\right]} = -\frac{0.5gN}{-0.5gN + g\left(1+\frac{N}{2}\right)^2 - \frac{GB}{T_0}} = \frac{0.5gN}{\left(1+\frac{N}{2}\right)^2 g - 0.5N_g - GC_p(1+N/2)}. \quad \text{(A8)}$$

Appendix 3

The equation used by Hysell et al. (2009, 2012) is

$$N^2 = -\frac{g}{\rho_0}\frac{\partial \rho_0}{\partial z} = \frac{g}{T}\left(\frac{\partial T}{\partial z} + \frac{g}{C_p}\right). \quad \text{(A1)}$$

Here, N^2 is the buoyancy frequency square and ρ_0 is the background density. This equation is incorrect because first, the buoyancy frequency for incompressible fluid is not equal to the buoyancy frequency for compressible fluid, and second, the background density given by the equation

$$\frac{1}{\rho_0}\frac{\partial\rho_0}{\partial z} = -\frac{1}{T}\left(\frac{\partial T}{\partial z} + \frac{g}{C_p}\right) \tag{A2}$$

is much larger than the density given by the equation

$$\frac{1}{\rho_A}\frac{\partial\rho_A}{\partial z} = -\frac{1}{T}\left(\frac{\partial T}{\partial z} + \frac{g}{R}\right) \tag{A3}$$

for hydrostatic equilibrium corresponding to real atmospheric conditions. For example, the scale height of the density is $H = \kappa T(1 + N/2)/mg$ corresponding to equation (A2) where $\partial T/\partial z = 0$ is larger by a factor of 3.5 than the scale height of the background atmospheric density $H = \kappa T/mg$ corresponding to equation (A3). The atmospheric density inferred from equation (A2) with $\partial T/\partial z = G$ is given by the formula

$$\rho_A = \rho_{A0}\{[T_{A0} + G(z - z_0)]/T_{A0}\}^{(-mg/\kappa G(1+0.5N)-1)} . \tag{A4}$$

This formula is similar to formula (13b) but with $G > 0$ and $-mg/\kappa G(1 + 0.5N)$ instead of $-mg/\kappa G$. The density given by formula (A4) is much larger than the density given by formula (13b) for $G > 0$. Substituting formula (A4) instead of the exponential term in equation (16) and using L'Hospital's rule, it is possible to get the equation

$$Ri_c = \frac{0.5gN}{g(1+0.5N)-0.5gN+GC_p(1+0.5N)} = \frac{0.5gN}{g+GC_p(1+0.5N)} \tag{A5}$$

instead of equation (17).

According to Fig. 2 in Hysell et al. (2012), a sporadic *E* layer with significant irregularities was observed by Arecibo INR at a height of around 110 km at 19:30 – 20:30 LT on July 2, 2010 in the lower thermosphere. The authors used the data on this layer to infer the parameters of the wind shear and then, using a numerical model, they estimated the Ri_c value of 0.75 for the dynamic instability corresponding to the observed irregularities in this region. According to the data shown in Fig. 2 (Hysell et al., 2012), the temperature gradient in the instability at around 110 km is G = 6-8 K/km and the Ri_c value can be found to be 0.8 – 0.65, respectively, according to equation (A5). It follows that the large Ri_c value of 0.75 estimated by the numerical model of Hysell et al. (2012) can only result from the large density used instead of the correct background density. In this case, the Ri_c value does not depend on the specific features of wind shear inferred by the authors and used in the numerical model. According to equation (17) with $G > 0$ and the background density given by formula (13b) with $G > 0$, the Ri_c value decreases from 0.25 to 0.2 with G increasing from 0 to 8 K/km.

Appendix 4

Formula (13b) is the same as the well-know and commonly used formula [Banks and Kockarts, *Aeronomy*, part A, page 36, Academic Press, 1973]:

$$\rho = \rho_0 \left(H / H_o \right)^{-(1+\beta)/\beta} \quad \text{(A1)}$$

or [Whitten and Poppoff, *Fundamentals of Aeronomy*, page 71, John Wiley and Sons, Inc., 1971]

$$n = n_0 \left(1 + \frac{\alpha}{H_0} \right)^{-(1+\alpha)/\alpha} \quad \text{(A2)}$$

where $H = \kappa T/mg$, $\alpha = \beta = \partial H/\partial z = (\kappa/mg)\partial T/\partial z$ *and* $n = \rho/m$. By substituting these relations in formulas (A1) or (A2), formula (13b) can be obtained.

Derivation of formula (13b)

Substituting $P = \rho\kappa T/m$ in the hydrostatic equation

$\partial p/\partial z = -\rho g$, (A3)

this equation can be written as

$$\frac{\kappa T}{m}\frac{\partial \rho}{\partial z} + \frac{\kappa}{m}\rho\frac{\partial T}{\partial z} = -\rho g\,. \qquad \text{(A4)}$$

Dividing equation (A4) by $\rho\kappa T/m$ yields

$$\frac{1}{\rho}\frac{\partial \rho}{\partial z} + \frac{1}{T}\frac{\partial T}{\partial z} = -\frac{mg}{\kappa T} \qquad \text{(A5)}$$

and this equation with $T = T_0 - Gz$ can be written as

$$\frac{1}{\rho}\frac{\partial \rho}{\partial z} = \frac{G}{T_0 - Gz} - \frac{mg}{\kappa(T_0 - Gz)} \qquad \text{(A6)}$$

where $G = \partial T/\partial z$. The solution of this equation is

$$\rho = \rho_0\left(\frac{T_0 - Gz}{T_0}\right)^{mg/\kappa G - 1} \qquad \text{(A7)}$$

and is the same as formula (13b) in the text.

Competing Interests

The authors declare that they have no conflict of interest.

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