Ann. Geophys., 34, 1145–1158, 2016
www.ann-geophys.net/34/1145/2016/
doi:10.5194/angeo-34-1145-2016

Annales Geophysicae
Open Access

Modeling anisotropic Maxwell–Jüttner distributions: derivation and properties

George Livadiotis

Southwest Research Institute, San Antonio, Texas, USA

Correspondence to: George Livadiotis (glivadiotis@swri.edu)

Received: 28 July 2016 – Revised: 26 October 2016 – Accepted: 5 November 2016 – Published: 2 December 2016

Abstract. In this paper we develop a model for the anisotropic Maxwell–Jüttner distribution and examine its properties. First, we provide the characteristic conditions that the modeling of consistent and well-defined anisotropic Maxwell–Jüttner distributions needs to fulfill. Then, we examine several models, showing their possible advantages and/or failures in accordance to these conditions. We derive a consistent model, and examine its properties and its connection with thermodynamics. We show that the temperature equals the average of the directional temperature-like components, as it holds for the classical, anisotropic Maxwell distribution. We also derive the internal energy and Boltzmann–Gibbs entropy, where we show that both are maximized for zero anisotropy, that is, the isotropic Maxwell–Jüttner distribution.

Keywords. Electromagnetics (plasmas) – solar physics astrophysics and astronomy (magnetic fields) – space plasma physics (kinetic and MHD theory)

1 Introduction

The Maxwell–Boltzmann distribution describes classical non-interacting particles at thermal equilibrium. This distribution is generalized within the framework of special relativity, leading to the Maxwell–Jüttner distribution, named after Ferencz Jüttner (1911).

In the case of anisotropic temperature, the Maxwell distribution becomes anisotropic on the kinetic degrees of freedom, which contribute differently in the internal energy of the system (e.g., solar wind: Olsen and Leer, 1999; Feldman et al., 1975; Pilipp et al., 1987; Phillips and Gosling, 1990; Kasper et al., 2002; Matteini et al., 2007; Štverák et al., 2008 – magnetosphere: Pilipp and Morfil, 1976; Renuka and Viswanathan, 1978; Tsurutani et al., 1982; Sckopke et al., 1990; Gary, 1992; Bavassano Cattaneo et al., 2006; Nishino et al., 2007; Cai et al., 2008; Winglee and Harnett, 2016 – also see the corresponding formulations in Krall and Trivelpiece, 1973; Livadiotis and McComas, 2014a; Livadiotis, 2015).

Therefore, we have two main types of generalization of the Maxwell–Boltzmann distribution, that is, the relativistic and the anisotropic description. However, there is no consistent modeling for both the generalizations together, namely, the anisotropic relativistic case, and the formalism of the anisotropic Maxwell–Jüttner distribution remains unknown. It is important to note that such a model may not be unique. From our experience on kappa distributions, a well-defined and unique isotropic model may degenerate to several anisotropic models, each one, however, corresponding to different physical meaning (see Sect. 5 in Livadiotis, 2015).

The purpose of this paper is to develop the anisotropic Maxwell–Jüttner distribution and examine its properties. First, the paper provides the characteristic conditions that a consistent and well-defined anisotropic modeling of the Maxwell–Jüttner distribution needs to fulfill. Then, guided by these conditions, the paper derives such a consistent model and, finally, examines several basic properties related to thermodynamics, e.g., the behavior of the internal energy, temperature, and entropy.

Next, Sect. 2 shows briefly the derivation of the standard, isotropic Maxwell–Jüttner distribution. Then, Sect. 3 proceeds to model the anisotropic Maxwell–Jüttner distribution. First, we provide the characteristic conditions of such a consistent model, and then examine several models, showing their possible advantages and/or failures in accordance with these conditions; finally, we end up with the correct model. In Sect. 4, we examine the properties and thermodynamics of this model, e.g., the internal energy, the partition of temperature to its directional components, and the entropy. Finally, Sect. 5 summarizes the conclusions.

2 Isotropic Maxwell–Jüttner distribution

The Maxwell–Boltzmann (MB) distribution P_{M} describes the velocities $\boldsymbol{u}$ or the kinetic energy $\varepsilon=\frac{1}{2}m\boldsymbol{u}^2$ of the particles at thermal equilibrium, far from the limit of the speed of light, i.e.,

$$P_{\mathrm{MB}}(\boldsymbol{p};\theta)=(\pi m^2\theta^2)^{-\frac{1}{2}d}\cdot\exp\left(-\frac{\frac{1}{2m}\boldsymbol{p}^2}{k_{\mathrm{B}}T}\right),\quad \theta\equiv\sqrt{2k_{\mathrm{B}}T/m},\;\; u\ll c, \tag{1a}$$

or, in terms of the kinetic energy,

$$P_{\mathrm{MB}}(\varepsilon;\,T)=\frac{(k_{\mathrm{B}}T)^{-\frac{1}{2}d}}{\Gamma(\frac{1}{2}d)}\cdot\exp\left(-\frac{\varepsilon}{k_{\mathrm{B}}T}\right)\cdot\varepsilon^{\frac{1}{2}d-1},\quad \varepsilon\ll mc^2, \tag{1b}$$

where θ is the temperature in speed dimensions, called thermal speed, and d denotes the kinetic degrees of freedom of each particle. (Note that the temperature is defined in the fluid's rest frame, where the bulk speed u_{b} is zero. In the non-relativistic case, this can be shown by using $\varepsilon=\frac{1}{2}m(\boldsymbol{u}-\boldsymbol{u}_{\mathrm{b}})^2$.)

The relativistic generalization of Eq. (1a), that is, the Maxwell–Jüttner (MJ) distribution, is given by

$$P_{\mathrm{MJ}}(\gamma)\propto\gamma^2\beta(\gamma)\cdot e^{-\frac{\gamma}{\Theta}},\quad \Theta\equiv\frac{k_{\mathrm{B}}T}{E_0},\quad E_0\equiv mc^2, \tag{2}$$

where $\boldsymbol{\beta}\equiv\frac{\boldsymbol{u}}{c}$ and $\gamma(\beta)\equiv(1-\beta^2)^{-\frac{1}{2}}$. (Note that the inverse of the unitless temperature Θ is the relativistic coldness ζ, Rezzola and Zanotti, 2013.) This distribution (Eq. 2) can be derived as follows.

According to the relativistic formalism for the particle momentum and energy, we have

$$\boldsymbol{p}=mc\cdot\gamma(\beta)\cdot\boldsymbol{\beta},\quad E(\beta)=\gamma(\beta)\cdot E_0, \tag{3}$$

while the kinetic energy is given by $\varepsilon=E-E_0=(\gamma-1)\cdot E_0$.

The Boltzmann distribution of a Hamiltonian is $P_{\mathrm{MJ}}(H)\propto\exp\left[-\frac{H}{k_{\mathrm{B}}T}\right]$. In the absence of a potential energy, H is simply given by the particle energy E, thus,

$$P_{\mathrm{MJ}}(E)\propto\exp\left(-\frac{E}{k_{\mathrm{B}}T}\right)\propto\exp\left(-\frac{\gamma}{\Theta}\right). \tag{4a}$$

(Note that E is the sum of the kinetic ε and inertial energy E_0, $\frac{\varepsilon}{k_{\mathrm{B}}T}=\frac{\gamma-1}{\Theta}$.) Then, we include the d-dimensional density of states:

$$P_{\mathrm{MJ}}(\gamma)\propto p(\gamma)^{d-1}\frac{\mathrm{d}p(\gamma)}{\mathrm{d}\gamma}\cdot\exp\left(-\frac{\gamma}{\Theta}\right) \tag{4b}$$

so that

$$\begin{aligned}\int P_{\mathrm{MJ}}(\boldsymbol{p})\mathrm{d}p_1\ldots\mathrm{d}p_d&\propto\int\exp\left[-\frac{E(\boldsymbol{p})}{k_{\mathrm{B}}T}\right]\mathrm{d}p_1\ldots\mathrm{d}p_d\\&=\int\exp\left[-\frac{E(p,\Omega_d)}{k_{\mathrm{B}}T}\right]\mathrm{d}\Omega_d\,p^{d-1}\mathrm{d}p\\&=\int_{\Omega_d}\exp\left[-\frac{E(\gamma,\Omega_d)}{k_{\mathrm{B}}T}\right]\cdot\left[p(\gamma)^{d-1}\frac{\mathrm{d}p(\gamma)}{\mathrm{d}\gamma}\right]\mathrm{d}\Omega_d\mathrm{d}\gamma,\end{aligned}$$

where Ω_d denotes the d-dimensional solid angle. For isotropic distributions, we have

$$\int P_{\mathrm{MJ}}(\boldsymbol{p})\mathrm{d}p_1\ldots\mathrm{d}p_d\propto\int_{\Omega_d}\exp\left[-\frac{E(\gamma)}{k_{\mathrm{B}}T}\right]\times\left[p(\gamma)^{d-1}\frac{\mathrm{d}p(\gamma)}{\mathrm{d}\gamma}\right]\mathrm{d}\Omega_d\mathrm{d}\gamma\equiv\int_{\Omega_d}\mathrm{d}\Omega_d\cdot\int P_{\mathrm{MJ}}(\gamma)\,\mathrm{d}\gamma, \tag{5a}$$

or

$$P_{\mathrm{MJ}}(\gamma)\propto\exp\left[-\frac{E(\gamma)}{k_{\mathrm{B}}T}\right]\cdot p(\gamma)^{d-1}\frac{\mathrm{d}p(\gamma)}{\mathrm{d}\gamma}. \tag{5b}$$

Then, $\mathrm{d}(\gamma\beta)=\gamma(\gamma^2-1)^{-\frac{1}{2}}\mathrm{d}\gamma=\beta^{-1}\mathrm{d}\gamma$ so that

$$p(\gamma)^{d-1}\frac{\mathrm{d}p(\gamma)}{\mathrm{d}\gamma}=(mc)^d(\gamma\beta)^{d-1}\frac{\mathrm{d}(\gamma\beta)}{\mathrm{d}\gamma}=(mc)^d\gamma^{d-1}\beta^{d-2}, \tag{6}$$

or

$$P_{\mathrm{MJ}}(\gamma)\propto\gamma^{d-1}\beta^{d-2}\cdot e^{-\frac{\gamma}{\Theta}}\propto\gamma(\gamma^2-1)^{\frac{d}{2}-1}\cdot e^{-\frac{\gamma}{\Theta}} \tag{7}$$

because $\frac{E}{k_{\mathrm{B}}T}=\frac{\gamma}{\Theta}$. Then, we normalize the distribution Eq. (7). We set

$$P_{\mathrm{MJ}}(\boldsymbol{p};\,\Theta)\,\mathrm{d}p_1\mathrm{d}p_2\ldots\mathrm{d}p_d=N\cdot e^{-\frac{\gamma(p)}{\Theta}}\,\mathrm{d}p_1\mathrm{d}p_2\ldots\mathrm{d}p_d, \tag{8}$$

and the angular integration,

$$\mathrm{d}p_1\mathrm{d}p_2\ldots\mathrm{d}p_d=B_d\,p^{d-1}\mathrm{d}p=\frac{1}{2}B_d(mc)^d\left[\left(\frac{p}{mc}\right)^2\right]^{\frac{d}{2}-1}d\left(\frac{p}{mc}\right)^2,$$

where $B_d=2\pi^{\frac{d}{2}}/\Gamma\left(\frac{d}{2}\right)$ is the surface of the unit d-dimensional sphere. Then, using the identity $\gamma^2=\left(\frac{p}{mc}\right)^2+1$, we have

$$P_{\mathrm{MJ}}(\boldsymbol{p}\,;\Theta)\,\mathrm{d}p_1\mathrm{d}p_2\ldots\mathrm{d}p_d=N\cdot\frac{1}{2}B_d(mc)^d\cdot e^{-\frac{\gamma}{\Theta}}\,(\gamma^2-1)^{\frac{d}{2}-1}d(\gamma^2-1), \tag{9}$$

and

$$1 = \int_{-\infty}^{\infty} P_{\mathrm{MJ}}(\boldsymbol{p}\,;\Theta)\mathrm{d}p_1\mathrm{d}p_2\ldots\mathrm{d}p_d \tag{10}$$

$$= N\cdot\frac{1}{2}B_d(mc)^d\cdot\int_0^{\infty} e^{-\frac{\gamma}{\Theta}}\,(\gamma^2-1)^{\frac{d}{2}-1}\mathrm{d}(\gamma^2-1)$$

$$= N\cdot\frac{1}{2}B_d\left(\frac{d}{2}\right)^{-1}(mc)^d\,\Theta^{-1}\cdot\int_1^{\infty} e^{-\frac{\gamma}{\Theta}}\,(\gamma^2-1)^{\frac{d}{2}}\mathrm{d}\gamma$$

$$= N\cdot\frac{1}{2}B_d\left(\frac{d}{2}\right)^{-1}(mc)^d\,\Theta^{-1}\cdot I_d,$$

where we have defined the integral

$$I_d \equiv \int_1^{\infty} e^{-\frac{\gamma}{\Theta}}\,(\gamma^2-1)^{\frac{d}{2}}\mathrm{d}\gamma. \tag{11}$$

The Macdonald function (Abramowitz and Stegun, 1972, p. 376) is defined by

$$K_n(z) \equiv \frac{\pi^{\frac{1}{2}}\left(\frac{1}{2}z\right)^n}{\Gamma\left(n+\frac{1}{2}\right)}\int_1^{\infty} e^{-z\gamma}\,(\gamma^2-1)^{n-\frac{1}{2}}\mathrm{d}\gamma \tag{12}$$

so that, by setting $n=\frac{d+1}{2}$, $z=\frac{1}{\Theta}$, we obtain

$$I_d = \Gamma\left(\frac{d}{2}+1\right)\pi^{-\frac{1}{2}}\cdot K_{\frac{d+1}{2}}\left(\frac{1}{\Theta}\right)(2\Theta)^{\frac{d+1}{2}}. \tag{13}$$

Hence,

$$N^{-1} = \frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}\left(\frac{d}{2}\right)^{-1}\Gamma\left(\frac{d}{2}+1\right)\pi^{-\frac{1}{2}}\cdot K_{\frac{d+1}{2}}\left(\frac{1}{\Theta}\right)(2\Theta)^{\frac{d+1}{2}}$$

$$= \pi^{\frac{d-1}{2}}2^{\frac{d+1}{2}}(mc)^d\cdot\Theta^{\frac{d-1}{2}}K_{\frac{d+1}{2}}\left(\frac{1}{\Theta}\right), \tag{14a}$$

or

$$N = \pi^{\frac{1-d}{2}}2^{-\frac{d+1}{2}}(mc)^{-d}\cdot\Theta^{\frac{1-d}{2}}K_{\frac{d+1}{2}}\left(\frac{1}{\Theta}\right)^{-1}. \tag{14b}$$

The inverse of the normalization constant gives the partition function $Z\equiv\frac{1}{N}$:

$$Z = \pi^{\frac{d-1}{2}}2^{\frac{d+1}{2}}(mc)^d\cdot\Theta^{\frac{d-1}{2}}K_{\frac{d+1}{2}}\left(\frac{1}{\Theta}\right). \tag{14c}$$

Therefore, the normalized distribution is

$$P_{\mathrm{MJ}}(\boldsymbol{p};\Theta)\;\mathrm{d}p_1\mathrm{d}p_2\ldots\mathrm{d}p_d \tag{15a}$$

$$= \pi^{\frac{1-d}{2}}2^{-\frac{d+1}{2}}(mc)^{-d}\cdot\Theta^{\frac{1-d}{2}}K_{\frac{d+1}{2}}\left(\frac{1}{\Theta}\right)^{-1}$$

$$\times\exp\left[-\gamma(\boldsymbol{p})/\Theta\right]\;\mathrm{d}p_1\mathrm{d}p_2\ldots\mathrm{d}p_d,$$

or we may derive the normalized distribution in terms of γ,

$$P_{\mathrm{MJ}}(\gamma\,;\Theta)\mathrm{d}\gamma \tag{15b}$$

$$= \frac{\pi^{\frac{1}{2}}2^{\frac{1-d}{2}}}{\Gamma\left(\frac{d}{2}\right)}\cdot\Theta^{\frac{1-d}{2}}K_{\frac{d+1}{2}}\left(\frac{1}{\Theta}\right)^{-1}\cdot e^{-\frac{\gamma}{\Theta}}\left(\gamma^2-1\right)^{\frac{d}{2}-1}\gamma\,\mathrm{d}\gamma.$$

(Note that in Sect. 4.4 we show that the parameter Θ coincides with the thermodynamic definition of temperature.)

In Fig. 1 we plot the MJ distribution of the kinetic energy $E_k=E_0(\gamma-1)$, for various temperatures and dimensionalities, which is given by $P\left(E_k\right)=P_{\mathrm{MJ}}\left(\gamma=1+\frac{E_k}{E_0}\right)$. We observe that the MJ distribution is well-approximated by the classical MB distribution, while the differences between the two distributions are more clear for high-energy and for temperature larger than $T\sim 1\,\mathrm{MeV}$.

When the MB distribution clearly deviates from the MJ distribution of the same temperature and dimensionality, then a different MB distribution can give a good approximation to the MJ distribution. This new MB distribution can be either (i) a convected MB distribution, that is, an MB distribution with the same dimensionality, but with different temperature T_{MB} and bulk speed u_{b} (or bulk energy $E_{\mathrm{b}}\equiv\frac{1}{2}mu_{\mathrm{b}}^2$), or (ii) an MB distribution with the same bulk speed, but with different temperature T_{MB} and degrees of freedom d_{MB}. These two types of approximations of the MJ distribution by an MB distribution are illustrated in Fig. 2.

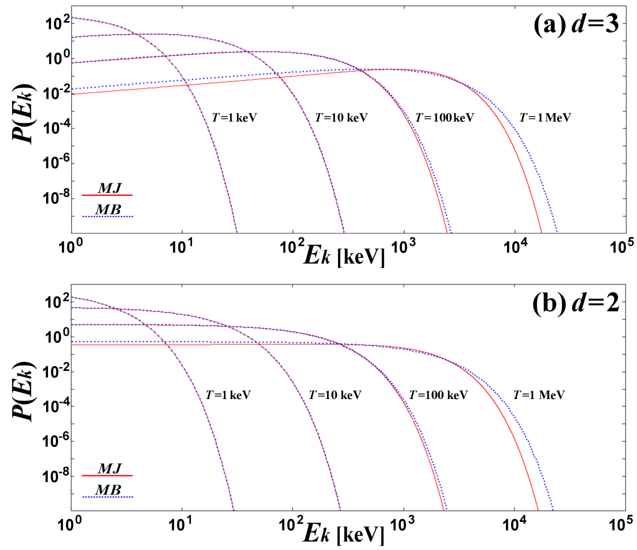

Figure 1. The MJ distribution of the electron kinetic energy $E_k=E_0(\gamma-1)$ with $E_0=511\,\mathrm{keV}$, plotted for various temperatures $T=1\,\mathrm{keV}$–$1\,\mathrm{MeV}$, and for dimensionality **(a)** $d=3$ and **(b)** $d=2$. For distribution values $>10^{-9}$ and temperatures less than $T\sim 10\,\mathrm{keV}$, the MJ distribution is well-approximated by the classical MB distribution, while the differences between the two distributions are more clear for $T\sim 1\,\mathrm{MeV}$.

Also useful is the expression of the distribution in the velocity space (Dunkel et al., 2007). Given that $\frac{\mathrm{d}(\beta\gamma)}{\mathrm{d}\beta} = \gamma^3$, we find

$$\begin{aligned}\mathrm{d}p_1 \ldots \mathrm{d}p_d &= p^{d-1}\mathrm{d}p\mathrm{d}\Omega_d \\ &= (mc)^d \gamma^{d-1}\beta^{d-1}\frac{\mathrm{d}(\beta\gamma)}{\mathrm{d}\beta}\mathrm{d}\beta\mathrm{d}\Omega_d \\ &= (mc)^d \gamma^{d+2}\beta^{d-1}\mathrm{d}\beta\mathrm{d}\Omega_d \\ &= (mc)^d \gamma^{d+2}\mathrm{d}\beta_1 \ldots \mathrm{d}\beta_d,\end{aligned}$$

hence

$$\begin{aligned}&P_{\mathrm{MJ}}(\boldsymbol{\beta};\Theta)\ \mathrm{d}\beta_1\mathrm{d}\beta_2\ldots\mathrm{d}\beta_d \qquad (15\mathrm{c})\\ &= \pi^{\frac{1-d}{2}} 2^{-\frac{d+1}{2}} \cdot \Theta^{\frac{1-d}{2}} K_{\frac{d+1}{2}}\left(\frac{1}{\Theta}\right)^{-1} \\ &\times \exp\left[-\frac{\gamma(\beta)}{\Theta}\right] \gamma(\beta)^{d+2}\mathrm{d}\beta_1\mathrm{d}\beta_2\ldots\mathrm{d}\beta_d.\end{aligned}$$

For example, for $d = 3$ we have

$$\begin{aligned}&P_{\mathrm{MJ}}(\boldsymbol{p};\Theta)\ \mathrm{d}p_1\mathrm{d}p_2\mathrm{d}p_3 \qquad (16\mathrm{a})\\ &= \frac{1}{4\pi}(mc)^{-3} \cdot \frac{1}{\Theta}K_2\left(\frac{1}{\Theta}\right)^{-1} \cdot e^{-\frac{\gamma(\boldsymbol{p})}{\Theta}}\mathrm{d}p_1\mathrm{d}p_2\mathrm{d}p_3\end{aligned}$$

and

$$\begin{aligned}&P_{\mathrm{MJ}}(\gamma;\Theta)\mathrm{d}\gamma = \qquad (16\mathrm{b})\\ &\frac{1}{\Theta}K_2\left(\frac{1}{\Theta}\right)^{-1} \cdot e^{-\frac{\gamma}{\Theta}}\left(\gamma^2-1\right)^{\frac{1}{2}}\gamma\mathrm{d}\gamma.\\ &P_{\mathrm{MJ}}(\boldsymbol{\beta};\Theta)\ \mathrm{d}\beta_1\mathrm{d}\beta_2\mathrm{d}\beta_3 = \qquad (16\mathrm{c})\\ &\frac{4}{\pi}\cdot\frac{1}{\Theta}K_2\left(\frac{1}{\Theta}\right)^{-1} \cdot \exp\left[-\frac{\gamma(\beta)}{\Theta}\right]\gamma(\beta)^5\mathrm{d}\beta_1\mathrm{d}\beta_2\mathrm{d}\beta_3.\end{aligned}$$

Note that there were several other attempts of further generalization of the MJ distribution, for example, using a power-law of energy, that is, $P(\gamma)\mathrm{d}\gamma \propto e^{-\frac{\gamma}{\Theta}}\left(\gamma^2-1\right)^{\frac{1}{2}}\gamma^{1-\eta}\mathrm{d}\gamma$ (e.g., see Horwitz et al., 1981; Lehmann, 2006; Dunkel et al., 2007); in this paper we focus on the standard MJ distribution.

3 Anisotropic Maxwell–Jüttner distribution

3.1 General characteristic conditions for consistent modeling

The derived distribution must have the following characteristic conditions.

1. Correspondence at both classical and isotropic limits:

   a. Correspondence for $c \to \infty$ (anisotropic MB).

   b. Correspondence for $\frac{T_i}{T} \to 1$ (isotropic MJ).

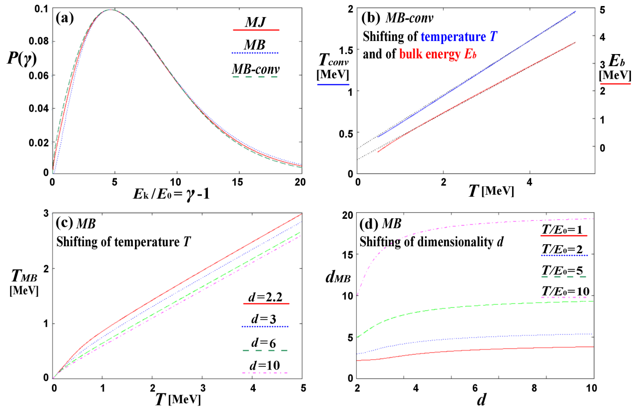

Figure 2. The plot of an MJ distribution of temperature T and degrees of freedom d can be misinterpreted either by (i) a convected MB distribution, that is, with the same dimensionality, but with different temperature T_{MB} and bulk speed u_{b} (or energy $E_{\mathrm{b}} \equiv \frac{1}{2}m\,u_{\mathrm{b}}^2$) or (ii) an MB distribution with the same bulk speed, but with different temperature T_{MB} and degrees of freedom d_{MB}. **(a)** A co-plot of an MJ distribution and the two mentioned MB distributions. **(b)** The temperature T_{MB} and the bulk energy E_{b} are plotted as functions of T. **(c)** The temperature T_{MB} as a function of T, for various dimensionalities, and **(d)** the dimensionality d_{MB} as a function of the dimensionality d and various temperatures.

2. Non-symplectic energy–temperature components:

   a. The inertial mass energy does not depend on kinetic terms and is isotropic.

   b. Each kinetic component is connected with the corresponding temperature component.

3. Temperature partition to its anisotropy components:

   a. Internal energy must be additive to its components, $U = \frac{d}{2}k_{\mathrm{B}}T$, $T = \frac{1}{d}\sum_{i=1}^{d}T_i$.

   b. Internal energy and entropy is maximized for zero anisotropy.

Next, we are going to derive and criticize several models using the above conditions.

3.2 Model: square inverse temperature

The particle energy–momentum relation can be a motive for a certain model of the anisotropic MJ distribution. Indeed, following the derivation of the energy–momentum relation,

$$\begin{aligned}&\left(\frac{1}{c}E,\boldsymbol{p}\right)^t \cdot \mathrm{diag}(+1,-1,-1,-1)\cdot\left(\frac{1}{c}E,\boldsymbol{p}\right) \qquad (17)\\ &= (mc,0)^t\cdot(mc,0) \Rightarrow E^2 - c^2\boldsymbol{p}^2 = (mc^2)^2,\end{aligned}$$

we may write

$$\left(\frac{1}{c}E,\boldsymbol{p}\right)^t\cdot\mathrm{diag}\left[+\left(\frac{1}{c}k_\mathrm{B}T\right)^{-2},-\left(\frac{1}{c}k_\mathrm{B}T_x\right)^{-2},\right. \tag{18}$$
$$\left.-\left(\frac{1}{c}k_\mathrm{B}T_y\right)^{-2},-\left(\frac{1}{c}k_\mathrm{B}T_z\right)^{-2}\right]\cdot\left(\frac{1}{c}E,\boldsymbol{p}\right)$$
$$=\left(\frac{E}{k_\mathrm{B}T}\right)^2-\left(\frac{p_xc}{k_\mathrm{B}T_x}\right)^2-\left(\frac{p_yc}{k_\mathrm{B}T_y}\right)^2-\left(\frac{p_zc}{k_\mathrm{B}T_z}\right)^2$$
$$=\left(\frac{mc^2}{k_\mathrm{B}T}\right)^2,$$

where the inertial energy is kept isotropic; i.e., the anisotropy characterizes only the momenta (according to the conditions in Sect. 3.1). Then, we may write Eq. (18) as

$$E^2-\sum_{i=1}^{3}\left(\frac{T}{T_i}\right)^2c^2p_i^2=(mc^2)^2,\ \text{or}, \tag{19}$$
$$\frac{E}{k_\mathrm{B}T}=\Theta^{-1}\cdot\sqrt{1+\sum_{i=1}^{3}\eta_i^2\left(\frac{p_i}{mc}\right)^2},$$

where $\eta_i\equiv\frac{T}{T_i}$ stands for the directional anisotropy.

The problem with the above derivation is that we are violating the energy–momentum relation of the free particle, $E(\boldsymbol{p})=\sqrt{c^2\boldsymbol{p}^2+(mc^2)^2}$, or $E(\boldsymbol{p})\cong\frac{\boldsymbol{p}^2}{2m}$ (for $\beta\ll1$), which constitute isotropic relations. Features from statistical mechanics should not be involved in one-particle (non-statistical) relations; in contrast, one-particle relations can certainly be examined by statistical mechanics. Nevertheless, we may obtain the same relation under different derivation path and physical interpretation.

The kinetic degrees of freedom, $(d\cdot N)=\sum_{m=1}^{M}(d\cdot N)_m$, are separated into M groups (particle subsystems of different temperature-like components): $(d\cdot N)_1$ of them have temperature component T_1 (group $m=1$), $(d\cdot N)_2$ of them have temperature component T_2 (group $m=2$), …, $(d\cdot N)_M$ of them have temperature component T_M (group $m=M$); then, the probability of the Hamiltonian factor becomes

$$\exp\left(-\sum_{i=1}^{d\cdot N}\frac{E_i}{k_\mathrm{B}T}\right)\to\exp\left(-\sum_{i_1=1}^{(d\cdot N)_1}\frac{E_{i_1}}{k_\mathrm{B}T_1}\right) \tag{20a}$$
$$\times\exp\left(-\sum_{i_2=1}^{(d\cdot N)_2}\frac{E_{i_2}}{k_\mathrm{B}T_2}\right)\cdot\ \ldots\ \cdot\exp\left(-\sum_{i_M=1}^{(d\cdot N)_M}\frac{E_{i_M}}{k_\mathrm{B}T_M}\right),$$

where E_i constitutes the energy component that depends solely on the ith kinetic degree of freedom. The temperature-like components do not constitute the unique temperature of the system that is given by T; they may be called temperatures though, for simplicity. Typically, the particle dimensionality determines the number of the groups, $M\le d$. Some examples are, when each dimension has a different temperature, $M=d$,

$$\exp\left(-\sum_{i=1}^{d\cdot N}\frac{E_i}{k_\mathrm{B}T}\right)\to\exp\left(-\sum_{i_1=1}^{N}\frac{E_{i_1}}{k_\mathrm{B}T_1}\right) \tag{20b}$$
$$\times\exp\left(-\sum_{i_2=1}^{N}\frac{E_{i_2}}{k_\mathrm{B}T_2}\right)\cdot\ \ldots\ \cdot\exp\left(-\sum_{i_d=1}^{N}\frac{E_{i_d}}{k_\mathrm{B}T_d}\right),$$

or when we have anisotropy at one direction, e.g., $d=3$, where one degree of freedom is parallel to a certain direction (e.g., that of the magnetic field), and two degrees of freedom are perpendicular to that, i.e.,

$$\exp\left(-\sum_{i=1}^{3N}\frac{E_i}{k_\mathrm{B}T}\right) \tag{21}$$
$$\to\exp\left(-\sum_{i_\perp=1}^{2N}\frac{E_{i_\perp}}{k_\mathrm{B}T_\perp}\right)\cdot\exp\left(-\sum_{i_\mathrm{II}=1}^{N}\frac{E_{i_\mathrm{II}}}{k_\mathrm{B}T_\mathrm{II}}\right).$$

Therefore, each group is characterized by the same one-particle relation but different statistics. For instance, in the previous examples, the one-particle distribution will be, respectively,

$$P_\mathrm{MB}(E_1,\ldots,E_d;T_1,\ldots,T_d)\propto\exp\left(-\sum_{i=1}^{d}\frac{E_i}{k_\mathrm{B}T_i}\right)\ \text{and}$$
$$P_\mathrm{MB}(E_\perp,E_\mathrm{II};T_\perp,T_\mathrm{II})\propto\exp\left(-\frac{E_\perp}{k_\mathrm{B}T_\perp}-\frac{E_\mathrm{II}}{k_\mathrm{B}T_\mathrm{II}}\right). \tag{22}$$

The anisotropy applies by considering different statistics for each component (or group); hence, while the temperature T characterizes the whole particle system and its internal energy, different temperature-like components characterize each of the kinetic degrees of freedom (non-symplectic property in Sect. 3.1),

$$E=\sum_{i=1}^{d}E_i\Leftrightarrow\frac{E}{k_\mathrm{B}T}\to\sum_{i=1}^{d}\frac{E_i}{k_\mathrm{B}T_i}. \tag{23}$$

Namely, in the isotropic MB distribution the independence of the probabilities $P(E_i;T_i)\propto\exp\left[-\frac{E_i}{k_\mathrm{B}T_i}\right]$ leads to the summation of the partial energies E_i, but in the anisotropic case it leads to the summation of $\frac{E_i}{k_\mathrm{B}T_i}$, i.e.,

$$P_\mathrm{MB}(E_1,\ldots,E_d;T_1,\ldots,T_d)\propto\prod_{i=1}^{d}P(E_i;T_i) \tag{24}$$
$$=\prod_{i=1}^{d}\exp\left(\frac{E_i}{k_\mathrm{B}T_i}\right)=\exp\left(-\sum_{i=1}^{d}\frac{E_i}{k_\mathrm{B}T_i}\right).$$

In the relativistic case, the probabilities are not independent, so their product cannot lead to the joint distribution P_MB.

$$P_\mathrm{MB}(E_1,\ldots,E_d;T_1,\ldots,T_d)\neq N\cdot\prod_{i=1}^{d}P(E_i;T_i) \tag{25}$$

(N: normalization constant). The model of square inverse temperature in Eq. (18), involves a square summation, in contrast to the linear model in Eq. (23), i.e.,

$$E=\sqrt{\sum_{i=1}^{d}E_i^2}\Leftrightarrow\frac{E}{k_\mathrm{B}T}\rightarrow\sqrt{\sum_{i=1}^{d}\left(\frac{E_i}{k_\mathrm{B}T_i}\right)^2}\text{ or}$$

$$E\rightarrow\sqrt{E_0^2+\sum_{i=1}^{d}\left(\frac{T}{T_i}\right)^2\cdot p_i^2c^2}. \tag{26}$$

Let $E_0=mc^2$ be the inertial energy and $E_i=cp_i$ the ith momentum component expressed in energy units. Then, we may use the L_q-type summation (e.g., Livadiotis, 2007),

$$A_1\oplus_q A_2\oplus_q\ldots\oplus_q A_d=(A_1^q+A_2^q+\ldots+A_d^q)^{\frac{1}{q}} \tag{27}$$

$$=\left(\sum_{i=1}^{d}A_i^q\right)^{\frac{1}{q}}.$$

According to this, the energies $(E_0,E_1,\ldots,E_d)$ are combined under an L_q summation to give

$$\frac{E}{k_\mathrm{B}T}\rightarrow\left[\frac{E_0^q}{(k_\mathrm{B}T_0)^q}+\sum_{i=1}^{d}\frac{E_i^q}{(k_\mathrm{B}T_i)^q}\right]^{\frac{1}{q}}, \tag{28a}$$

that is,

$$\frac{E}{k_\mathrm{B}T}\rightarrow\frac{E_0}{k_\mathrm{B}T_0}+\sum_{i=1}^{d}\frac{E_i}{k_\mathrm{B}T_i},\text{ for }q=1,\text{ or} \tag{28b}$$

$$\frac{E}{k_\mathrm{B}T}\rightarrow\sqrt{\frac{E_0^2}{(k_\mathrm{B}T_0)^2}+\sum_{i=1}^{d}\frac{E_i^2}{(k_\mathrm{B}T_i)^2}}\text{ for }q=2.$$

Again, we reiterate that the inertial energy E_0 is common in all directions and it does not contribute to the thermal anisotropy so that its temperature directional component equals the total temperature, $T_0=T$,

$$P(E_0,E_1,\ldots,E_d;T,T_1,\ldots,T_d)$$

$$\propto\exp\left[-\sqrt{\frac{E_0^2}{(k_\mathrm{B}T)^2}+\sum_{i=1}^{d}\frac{E_i^2}{(k_\mathrm{B}T_i)^2}}\right] \tag{29}$$

$$=\exp\left[-\frac{E_0}{k_\mathrm{B}T}\cdot\sqrt{1+\sum_{i=1}^{d}\left(\frac{T}{T_iE_0}\right)^2\cdot E_i^2}\right].$$

(Note that the temperature T is a function of the directional temperature-like components T_i, thus, it may be excluded from the independent components of the probability distribution P; the same can be done for the inertial energy E_0, as it is not a variable for particles of a certain mass.) Hence, we write

$$P(\boldsymbol{p};\boldsymbol{T})\propto\exp\left[-\frac{E_0}{k_\mathrm{B}T}\cdot\sqrt{1+\sum_{i=1}^{d}\left(\frac{T}{T_i}\right)^2\cdot\left(\frac{p_i}{mc}\right)^2}\right], \tag{30}$$

where we use the vector-like notation $\boldsymbol{T}=(T_1,\ldots,T_d)$. The disproof of this formalism comes from the failure to recover the anisotropic MB distribution. Indeed,

$$P(\boldsymbol{p};\boldsymbol{T})\propto\exp\left[-\frac{E_0}{k_\mathrm{B}T}\cdot\sqrt{1+\sum_{i=1}^{d}\left(\frac{T}{T_i}\right)^2\cdot\left(\frac{p_i}{mc}\right)^2}\right]$$

$$\cong\exp\left[-\sum_{i=1}^{d}\left(\frac{T}{T_i}\right)\cdot\left(\frac{\frac{1}{2m}p_i^2}{k_\mathrm{B}T_i}\right)\right], \tag{31}$$

which differs from the expected

$$P(\boldsymbol{p};\boldsymbol{T})\propto\exp\left[-\sum_{i=1}^{d}\left(\frac{\frac{1}{2m}p_i^2}{k_\mathrm{B}T_i}\right)\right]. \tag{32}$$

3.3 Model: symplectic inverse temperature

It is always possible to write the anisotropic relation in a symplectic way so that both correspondence conditions hold (Sect. 3.1). In the following example, the inertial energy is characterized by anisotropy, while there is one symplectic kinetic component ($i=2$):

$$E=\sqrt{\sum_{i=0}^{2}E_i^2} \tag{33a}$$

$$\Leftrightarrow\frac{E}{k_\mathrm{B}T}\rightarrow\sqrt{\frac{E_0^2}{(k_\mathrm{B}T_1)^2}+\frac{E_1^2}{(k_\mathrm{B}T_1)^2}+\frac{E_2^2}{(k_\mathrm{B}T_1)\,(k_\mathrm{B}T_2)}},$$

and thus the distribution becomes

$$P(\boldsymbol{p};\boldsymbol{T}) \tag{33b}$$

$$\propto\exp\left[-\sqrt{\frac{E_0^2}{(k_\mathrm{B}T_1)^2}+\frac{(p_1c)^2}{(k_\mathrm{B}T_1)^2}+\frac{(p_2c)^2}{(k_\mathrm{B}T_1)\,(k_\mathrm{B}T_2)}}\right].$$

Therefore, the correspondence conditions are fulfilled; i.e., the Hamiltonian factor is approximated for $c\rightarrow\infty$ to

$$P(\boldsymbol{p};\boldsymbol{T})\propto\exp\left[-\sum_{i=1}^{2}\left(\frac{\frac{1}{2m}p_i^2}{k_\mathrm{B}T_i}\right)\right], \tag{33c}$$

which is the anisotropic MB distribution. However, the symplectic property is not fulfilled (Sect. 3.1).

In another example, for any value of b, we model the Hamiltonian factor as

$$E=\sqrt{\sum_{i=0}^{2}E_i^2}\Leftrightarrow\frac{E}{k_\mathrm{B}T}\rightarrow\sqrt{\frac{E_0^2}{(k_\mathrm{B}T)^2\left(T_1T_2/T^2\right)^{2b}}+\frac{E_1^2}{(k_\mathrm{B}T_1)\,(k_\mathrm{B}T)\left(T_1T_2/T^2\right)^b}+\frac{E_2^2}{(k_\mathrm{B}T_2)\,(k_\mathrm{B}T)\left(T_1T_2/T^2\right)^b}}, \tag{34a}$$

and thus the distribution becomes

$$P(\boldsymbol{p};\boldsymbol{T}) \propto \exp\left[-\sqrt{\frac{E_0^2}{(k_{\mathrm{B}}T)^2\left(T_1T_2/T^2\right)^{2b}} + \frac{(p_1c)^2}{(k_{\mathrm{B}}T_1)(k_{\mathrm{B}}T)\left(T_1\,T_2/T^2\right)^b} + \frac{(p_2c)^2}{(k_{\mathrm{B}}T_2)(k_{\mathrm{B}}T)\left(T_1\,T_2/T^2\right)^b}}\right]. \quad (34b)$$

The Hamiltonian factor is approximated for $c \to \infty$ to

$$\sqrt{\frac{E_0^2}{(k_{\mathrm{B}}T)^2\left(T_1\,T_2/T^2\right)^{2b}} + \frac{(p_1c)^2}{(k_{\mathrm{B}}T_1)(k_{\mathrm{B}}T)\left(T_1\,T_2/T^2\right)^b} + \frac{(p_2c)^2}{(k_{\mathrm{B}}T_2)(k_{\mathrm{B}}T)\left(T_1\,T_2/T^2\right)^b}} \cong \frac{E_0}{(k_{\mathrm{B}}T)\left(T_1\,T_2/T^2\right)^b} + \frac{\frac{1}{2m}p_1^2}{k_{\mathrm{B}}T_1} + \frac{\frac{1}{2m}p_2^2}{k_{\mathrm{B}}T_2} \quad (34c)$$

so that

$$P(\boldsymbol{p};\boldsymbol{T}) \propto \exp\left[-\sum_{i=1}^{2}\left(\frac{\frac{1}{2m}p_i^2}{k_{\mathrm{B}}T_i}\right)\right]. \quad (34d)$$

For example, for $b = \frac{1}{2}$, we obtain

$$P(\boldsymbol{p};\boldsymbol{T}) \propto \exp\left[-\sqrt{\frac{E_0^2}{(k_{\mathrm{B}}T_1)(k_{\mathrm{B}}T_2)} + \frac{(p_1c)^2}{(k_{\mathrm{B}}T_1)^{\frac{3}{2}}(k_{\mathrm{B}}T_2)^{\frac{1}{2}}} + \frac{(p_2c)^2}{(k_{\mathrm{B}}T_1)^{\frac{1}{2}}(k_{\mathrm{B}}T_2)^{\frac{3}{2}}}}\right]. \quad (35)$$

For $b = 0$, we derive the only anisotropic distribution that obeys the correspondence and also the symplectic conditions in Sect. 3.1, which is examined as follows:

$$P(\boldsymbol{p};\boldsymbol{T}) \propto \exp\left[-\sqrt{\frac{E_0^2}{(k_{\mathrm{B}}T)^2} + \frac{(p_1c)^2}{(k_{\mathrm{B}}T)(k_{\mathrm{B}}T_1)} + \frac{(p_2c)^2}{(k_{\mathrm{B}}T)(k_{\mathrm{B}}T_2)}}\right]. \quad (36)$$

Let us now suggest a model that fulfills all the requirements given in Sect. 3.1.

3.4 Suggested model: linear inverse temperature

We may write the energy–momentum relation in a way that the summation on the energy component is linear, namely,

$$E = \sqrt{E_0^2 + \sum_{i=1}^{d} c^2 p_i^2} = \sqrt{E_0 \cdot \left(E_0 + \sum_{i=1}^{d} \frac{p_i^2}{m}\right)} = \sqrt{E_0 \cdot \left(E_0 + \sum_{i=1}^{d} E_i\right)}. \quad (37a)$$

Hence, the Hamiltonian factor becomes

$$\frac{E}{k_{\mathrm{B}}T} \to \sqrt{\frac{E_0}{k_{\mathrm{B}}T} \cdot \left(\frac{E_0}{k_{\mathrm{B}}T} + \sum_{i=1}^{d} \frac{E_i}{k_{\mathrm{B}}T_i}\right)}, \quad (37b)$$

where $E_0 = mc^2$ and $E_i = \frac{1}{m}p_i^2$. Then, the distribution becomes

$$P(\boldsymbol{p};\boldsymbol{T}) \propto \exp\left[-\frac{E_0}{k_{\mathrm{B}}T} \cdot \sqrt{1 + \sum_{i=1}^{d} \frac{T}{T_i}\frac{E_i}{E_0}}\right] = \exp\left[-\frac{E_0}{k_{\mathrm{B}}T} \cdot \sqrt{1 + \sum_{i=1}^{d} \frac{T}{T_i}\left(\frac{p_i}{mc}\right)^2}\right]. \quad (37c)$$

– Correspondence for $c \to \infty$ (anisotropic MB):

$$P(\boldsymbol{p};\boldsymbol{T}) \propto \exp\left[-\frac{E_0}{k_{\mathrm{B}}T} \cdot \sqrt{1 + \sum_{i=1}^{d} \frac{T}{T_i}\left(\frac{p_i}{mc}\right)^2}\right] \cong \exp\left(-\frac{E_0}{k_{\mathrm{B}}T} - \sum_{i=1}^{d} \frac{\frac{1}{2m}p_i^2}{k_{\mathrm{B}}T_i}\right) \propto \exp\left(-\sum_{i=1}^{d} \frac{\frac{1}{2m}p_i^2}{k_{\mathrm{B}}T_i}\right).$$

– Correspondence for $T_i/T \to 1$ (isotropic MJ):

$$P(\boldsymbol{p};T) \propto \exp\left[-\frac{E_0}{k_{\mathrm{B}}T} \cdot \sqrt{1 + \left(\frac{p}{mc}\right)^2}\right] \propto \exp\left(-\frac{\gamma}{\Theta}\right).$$

The normalized anisotropic Maxwell–Jüttner distribution is

$$P(\boldsymbol{p};\boldsymbol{T}) = N \cdot \exp\left[-\frac{E_0}{k_{\mathrm{B}}T} \cdot \sqrt{1 + \sum_{i=1}^{d} \frac{T}{T_i}\left(\frac{p_i}{mc}\right)^2}\right] \quad (38a)$$

or

$$P(\boldsymbol{p};\boldsymbol{\Theta}) = N \cdot \exp\left[-\frac{1}{\Theta} \cdot \sqrt{1 + \sum_{i=1}^{d} \frac{\Theta}{\Theta_i}\left(\frac{p_i}{mc}\right)^2}\right],$$

with normalization

$$N^{-1} = \int_{-\infty}^{\infty} \exp\left[-\frac{E_0}{k_{\mathrm{B}}T} \cdot \sqrt{1 + \sum_{i=1}^{d} \frac{T}{T_i}\left(\frac{p_i}{mc}\right)^2}\right] \mathrm{d}p_1 \ldots \mathrm{d}p_d$$
$$= B_d \left(\frac{E_0}{k_{\mathrm{B}}T}\right)^{\frac{1}{2}d} \prod_{i=1}^{d}\left(\sqrt{\frac{k_{\mathrm{B}}T_i}{E_0}}\right)(mc)^d \times \int_0^{\infty} e^{-\frac{E_0}{k_{\mathrm{B}}T}\cdot\sqrt{1+\widetilde{p}^2}}\,\widetilde{p}^{d-1}\mathrm{d}\widetilde{p}$$
$$= \frac{1}{2} B_d \left(\frac{E_0}{k_{\mathrm{B}}T}\right)^{\frac{1}{2}d} \prod_{i=1}^{d}\left(\sqrt{\frac{k_{\mathrm{B}}T_i}{E_0}}\right)(mc)^d$$

$$\times \int_1^\infty e^{-\frac{E_0}{k_B T}\cdot\widetilde{\gamma}}(\widetilde{\gamma}^2-1)^{\frac{1}{2}d-1}\mathrm{d}(\widetilde{\gamma}^2-1)$$

$$= \frac{1}{2}B_d\left(\frac{E_0}{k_B T}\right)^{\frac{1}{2}d+1}\prod_{i=1}^{d}\left(\sqrt{\frac{k_B T_i}{E_0}}\right)(mc)^d\cdot I_d,$$

where we set $\widetilde{\gamma}\equiv\sqrt{1+\widetilde{p}^2}$ and $\widetilde{p}^2\equiv\sum_{i=1}^{d}\left(\frac{T}{T_i}\right)\left(\frac{1}{mc}p_i\right)^2$. Hence, we have

$$N=\pi^{\frac{1-d}{2}}2^{-\frac{d+1}{2}}(mc)^{-d}\cdot\left(\frac{k_B T}{E_0}\right)^{\frac{1}{2}} \prod_{i=1}^{d}\left(\frac{k_B T_i}{E_0}\right)^{-\frac{1}{2}}K_{\frac{d+1}{2}}\left(\frac{E_0}{k_B T}\right)^{-1}. \tag{38b}$$

Note that for the specific case where $d=3$ and $\{T_i\}_{i=1}^d=(T_{\mathrm{II}},T_\perp)$ the normalization constant in Eq. (38b) coincides with that derived by Treumann and Baumjohann (2016). Then, the normalized distribution is

$$P(\boldsymbol{p};\boldsymbol{T})\mathrm{d}p_1\ldots\mathrm{d}p_d=\pi^{\frac{1-d}{2}}2^{-\frac{d+1}{2}}\cdot\left(\frac{k_B T}{E_0}\right)^{\frac{1}{2}} \prod_{i=1}^{d}\left(\frac{k_B T_i}{E_0}\right)^{-\frac{1}{2}}K_{\frac{d+1}{2}}\left(\frac{E_0}{k_B T}\right)^{-1}\cdot(mc)^{-d} \times\exp\left[-\frac{E_0}{k_B T}\cdot\sqrt{1+\sum_{i=1}^{d}\frac{T}{T_i}\left(\frac{p_i}{mc}\right)^2}\right]\mathrm{d}p_1\ldots\mathrm{d}p_d, \tag{39a}$$

or, in terms of Θ_i,

$$P(\boldsymbol{p};\boldsymbol{\Theta})\mathrm{d}p_1\ldots\mathrm{d}p_d=\pi^{\frac{1-d}{2}}2^{-\frac{d+1}{2}}\cdot\Theta^{\frac{1}{2}} \prod_{i=1}^{d}\Theta_i^{-\frac{1}{2}}K_{\frac{d+1}{2}}\left(\frac{E_0}{k_B T}\right)^{-1}\cdot(mc)^{-d} \times\exp\left[-\frac{1}{\Theta}\cdot\sqrt{1+\sum_{i=1}^{d}\frac{\Theta}{\Theta_i}\left(\frac{p_i}{mc}\right)^2}\right]\mathrm{d}p_1\ldots\mathrm{d}p_d. \tag{39b}$$

It is useful to write the distribution also in terms of the following arguments:

$$\widetilde{p}_i\equiv\frac{\Theta}{\Theta_i}\frac{p_i}{mc},\;\; \widetilde{p}^2\equiv\sum_{i=1}^{d}\widetilde{p}_i^2,\;\; \widetilde{\gamma}\equiv\sqrt{1+\widetilde{p}^2},\text{ thus,} \tag{40}$$

$$P\left(\widetilde{\boldsymbol{p}};\Theta\right)\mathrm{d}\widetilde{p}_1\ldots\mathrm{d}\widetilde{p}_d=\pi^{\frac{1-d}{2}}2^{-\frac{d+1}{2}}\cdot(mc)^{-d} \cdot\Theta^{\frac{1-d}{2}}K_{\frac{d+1}{2}}\left(\frac{1}{\Theta}\right)^{-1}\cdot\exp\left[-\frac{\widetilde{\gamma}\left(\widetilde{\boldsymbol{p}}\right)}{\Theta}\right]\mathrm{d}\widetilde{p}_1\ldots\mathrm{d}\widetilde{p}_d, \tag{41a}$$

and

$$P(\widetilde{\gamma};\Theta)\mathrm{d}\widetilde{\gamma}=\frac{\pi^{\frac{1}{2}}2^{\frac{1-d}{2}}}{\Gamma\left(\frac{d}{2}\right)}\cdot\Theta^{\frac{1-d}{2}}K_{\frac{d+1}{2}}\left(\frac{1}{\Theta}\right)^{-1} \times\exp\left(-\frac{\widetilde{\gamma}}{\Theta}\right)\cdot\left(\widetilde{\gamma}^2-1\right)^{\frac{d}{2}-1}\widetilde{\gamma}\,\mathrm{d}\widetilde{\gamma}. \tag{41b}$$

– Correspondence for $T_i/T\to1$ (isotropic MJ):

$$\begin{aligned}P(p;\Theta)\mathrm{d}p&=\int_{\Omega_d}P(\boldsymbol{p};\Theta)\,\mathrm{d}p_1\ldots\mathrm{d}p_d\\&=B_d\,P(p;\Theta)\,p^{d-1}\mathrm{d}p\\&=\frac{\pi^{\frac{1}{2}}2^{\frac{1-d}{2}}}{\Gamma\left(\frac{d}{2}\right)}\Theta^{\frac{1-d}{2}}K_{\frac{d+1}{2}}\left(\frac{1}{\Theta}\right)^{-1}(mc)^{-d}\\&\quad\times\exp\left[-\frac{1}{\Theta}\cdot\sqrt{1+\left(\frac{p}{mc}\right)^2}\right]p^{d-1}\mathrm{d}p\\&=\frac{\pi^{\frac{1}{2}}2^{\frac{1-d}{2}}}{\Gamma\left(\frac{d}{2}\right)}\Theta^{\frac{1-d}{2}}K_{\frac{d+1}{2}}\left(\frac{1}{\Theta}\right)^{-1}\\&\quad\times e^{-\frac{\gamma}{\Theta}}\left(\gamma^2-1\right)^{\frac{d}{2}-1}\gamma\,\mathrm{d}\gamma,\end{aligned}$$

which coincides with the isotropic MJ distribution in Eq. (15b).

– Correspondence for $c\to\infty$ (anisotropic MB):

$$\begin{aligned}&\pi^{\frac{1-d}{2}}2^{-\frac{d+1}{2}}\cdot\left(\frac{k_B T}{E_0}\right)^{\frac{1}{2}}\prod_{i=1}^{d}\left(\frac{k_B T_i}{E_0}\right)^{-\frac{1}{2}}K_{\frac{d+1}{2}}\left(\frac{E_0}{k_B T}\right)^{-1}\\&\times(mc)^{-d}\cdot\exp\left[-\frac{E_0}{k_B T}\cdot\sqrt{1+\sum_{i=1}^{d}\frac{T}{T_i}\left(\frac{p_i}{mc}\right)^2}\right]\mathrm{d}p_1\ldots\mathrm{d}p_d\\&=2^{-\frac{1}{2}}\pi^{\frac{1-d}{2}}\cdot\left(\frac{k_B T}{E_0}\right)^{\frac{1}{2}}\prod_{i=1}^{d}\left(\sqrt{\frac{m}{2k_B T_i}}\right)\cdot K_{\frac{d+1}{2}}\left(\frac{E_0}{k_B T}\right)^{-1}\\&\times\exp\left[-\frac{E_0}{k_B T}\cdot\sqrt{1+\sum_{i=1}^{d}\frac{T}{T_i}(\gamma_i\beta_i)^2}\right]\mathrm{d}(\gamma_1 u_1)\ldots\mathrm{d}(\gamma_d u_d)\\&\cong\pi^{-\frac{d}{2}}\cdot\prod_{i=1}^{d}\left(\sqrt{\frac{m}{2k_B T_i}}\right)\cdot\sqrt{\frac{\pi}{2}\frac{k_B T}{E_0}}K_{\frac{d+1}{2}}\left(\frac{E_0}{k_B T}\right)^{-1}\\&\times\exp\left(-\frac{E_0}{k_B T}\right)\cdot\exp\left(-\sum_{i=1}^{d}\frac{m}{2k_B T_i}u_i^2\right)\mathrm{d}u_1\ldots\mathrm{d}u_d\\&\Rightarrow P(\boldsymbol{u};\boldsymbol{T})\mathrm{d}u_1\ldots\mathrm{d}u_d=\pi^{-\frac{d}{2}}\cdot\prod_{i=1}^{d}\theta_i^{-1}\\&\times\exp\left(-\sum_{i=1}^{d}\frac{u_i^2}{\theta_i^2}\right)\mathrm{d}u_1\ldots\mathrm{d}u_d,\end{aligned}$$

which is the anisotropic MB distribution, where $\sqrt{\frac{\pi}{2}\frac{k_B T}{E_0}}K_{\frac{d+1}{2}}\left(\frac{E_0}{k_B T}\right)^{-1}\exp\left(-\frac{E_0}{k_B T}\right)\cong1+O\left(\frac{k_B T}{E_0}\right)$. Figure 3 shows the cases of anisotropic/isotropic MJ and MB distributions.

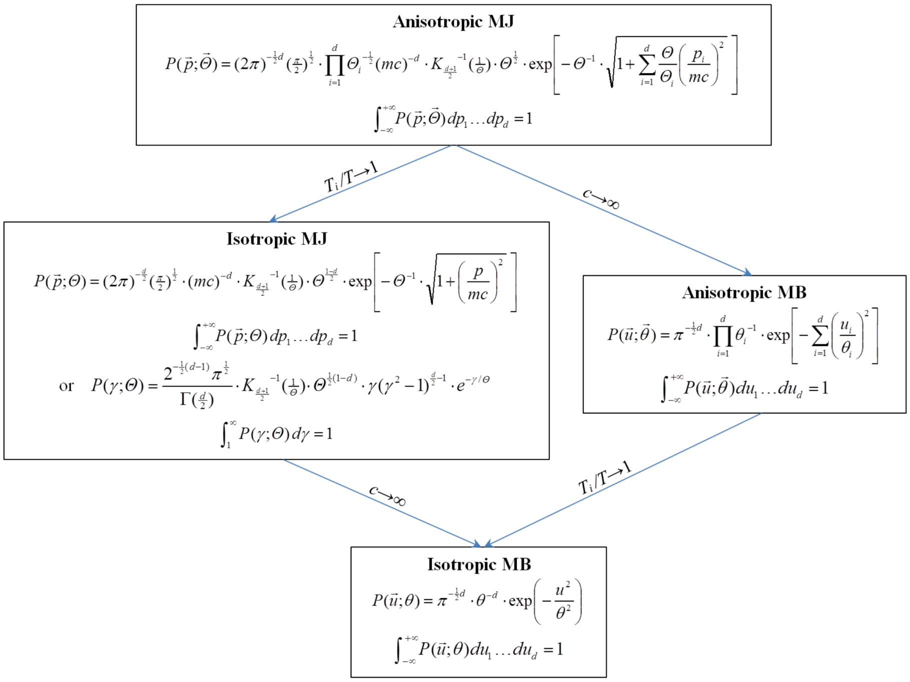

Figure 3. Correspondence of anisotropic Maxwell–Jüttner (MJ) distribution to the isotropic Maxwell–Jüttner and anisotropic Maxwell distributions; we use $\Theta_i \equiv \frac{k_B T_i}{E_0} = \frac{1}{2}\frac{\theta_i^2}{c^2}$, $\theta_i \equiv \sqrt{2\frac{k_B T_i}{m}}$. MB is the Maxwell–Boltzmann distribution.

4 Properties and thermodynamics

4.1 Internal energy

We start with the isotropic MJ distribution. The internal energy, U, is derived by the mean energy:

$$\begin{aligned}\frac{U}{E_0} &= \langle\gamma\rangle = \int_1^\infty P_{\mathrm{MJ}}(\gamma\,;\Theta)\gamma\,\mathrm{d}\gamma \\ &= \frac{\pi^{\frac{1}{2}}2^{\frac{1-d}{2}}}{\Gamma(\frac{d}{2})}\cdot\Theta^{\frac{1-d}{2}}K_{\frac{d+1}{2}}\left(\frac{1}{\Theta}\right)^{-1} \\ &\times\int_1^\infty e^{-\frac{\gamma}{\Theta}}\left(\gamma^2-1\right)^{\frac{d}{2}-1}\gamma^2\,\mathrm{d}\gamma \\ &= \frac{\pi^{\frac{1}{2}}2^{\frac{1-d}{2}}}{\Gamma\left(\frac{d}{2}\right)}\cdot\Theta^{\frac{1-d}{2}}K_{\frac{d+1}{2}}\left(\frac{1}{\Theta}\right)^{-1}\cdot(I_d+I_{d-2}),\end{aligned} \tag{42a}$$

where, according to Eq. (11), we have

$$\begin{aligned}I_d+I_{d-2} =&\Gamma\left(\frac{d}{2}\right)\pi^{-\frac{1}{2}}2^{\frac{d-1}{2}}\Theta^{\frac{d+1}{2}}K_{\frac{d+1}{2}}\left(\frac{1}{\Theta}\right) \\ &\times\left[d+\frac{\frac{1}{\Theta}K_{\frac{d-1}{2}}\left(\frac{1}{\Theta}\right)}{K_{\frac{d+1}{2}}\left(\frac{1}{\Theta}\right)}\right].\end{aligned} \tag{42b}$$

Hence,

$$\frac{U}{E_0} = d\cdot\Theta+\frac{K_{\frac{d-1}{2}}\left(\frac{1}{\Theta}\right)}{K_{\frac{d+1}{2}}\left(\frac{1}{\Theta}\right)}, \tag{42c}$$

or

$$\begin{aligned}U &= E_0+d\cdot k_B T+E_0\left[\frac{K_{\frac{d-1}{2}}\left(\frac{1}{\Theta}\right)}{K_{\frac{d+1}{2}}\left(\frac{1}{\Theta}\right)}-1\right] \\ &\cong E_0+\frac{d}{2}k_B T+k_B T\cdot O\left(\frac{k_B T}{E_0}\right),\end{aligned} \tag{43}$$

where $\frac{K_{\frac{d-1}{2}}\left(\frac{1}{\Theta}\right)}{K_{\frac{d+1}{2}}\left(\frac{1}{\Theta}\right)} \cong 1-\frac{d}{2}\Theta+O(\Theta^2)$.

In the anisotropic case, the corresponding integral cannot be solved analytically,

$$\begin{aligned}\frac{U}{E_0} &= \langle\gamma\rangle = \int_1^\infty P_{\mathrm{MJ}}(\widetilde{\gamma}\,;\Theta)\,\gamma\,\mathrm{d}\widetilde{\gamma} \\ &= \frac{\pi^{\frac{1}{2}}2^{\frac{1-d}{2}}}{\Gamma\left(\frac{d}{2}\right)}\Theta^{\frac{1-d}{2}}K_{\frac{d+1}{2}}\left(\frac{1}{\Theta}\right)^{-1} \\ &\times\int_1^\infty\sqrt{1+(\widetilde{\gamma}^2-1)\frac{1}{\Theta}\sum_{i=1}^d\Theta_i l_i}\cdot e^{-\frac{\widetilde{\gamma}}{\Theta}}\left(\widetilde{\gamma}^2-1\right)^{\frac{d}{2}-1}\widetilde{\gamma}\,\mathrm{d}\widetilde{\gamma},\end{aligned} \tag{44}$$

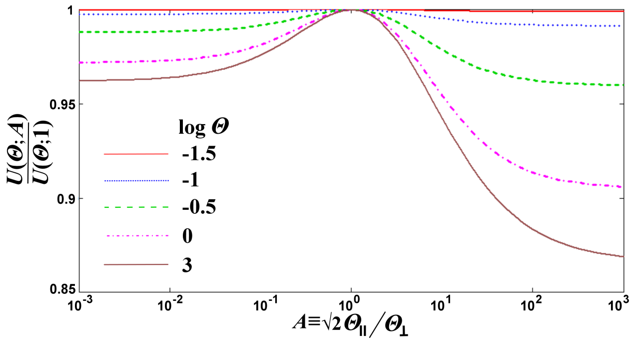

Figure 4. The internal energy depends on the anisotropy A, maximized in the case of isotropic temperature, $A=0$. At the classical limit of $\Theta \to 0$, i.e., for $c \to \infty$, the internal energy becomes independent of the temperature directional components Θ_i.

where $l_i = \left(\frac{\tilde{p}_i}{\tilde{p}}\right)^2$ is the square of the direction cosine at the ith axis. For example, for $d=3$, we have $l_x = \sin^2\vartheta \cos^2\phi$, $l_y = \sin^2\vartheta \sin^2\phi$, $l_z = \cos^2\vartheta$. Hence,

$$\frac{U}{E_0} = \Theta^{-\frac{3}{2}} K_2\left(\frac{1}{\Theta}\right)^{-1} \int_1^{\infty} [\ldots] \mathrm{d}\tilde{\gamma} \mathrm{d}\cos\vartheta \mathrm{d}\phi \tag{45a}$$

$$[\ldots] \equiv \sqrt{\Theta + \left(\tilde{\gamma}^2 - 1\right)\left(\Theta_z \cos^2\vartheta + \sin^2\vartheta \cdot \left(\Theta_x \cos^2\phi + \Theta_y \sin^2\phi\right)\right)} \cdot e^{-\frac{\tilde{\gamma}}{\Theta}} \left(\tilde{\gamma}^2 - 1\right)^{\frac{1}{2}} \tilde{\gamma}.$$

In the case of the parallel/perpendicular anisotropy, we obtain

$$\frac{1}{E_0} U\left(\Theta_\perp, \Theta_{\rm II}\right) = \frac{3K_2\left(\frac{3}{\Theta_{\rm II}+2\Theta_\perp}\right)^{-1}}{\left(\Theta_{\rm II}+2\Theta_\perp\right)^{\frac{3}{2}}} \tag{45b}$$

$$\times \int_1^{\infty} \sqrt{\frac{1}{3}\left(\Theta_{\rm II}+2\Theta_\perp\right) + \left(\tilde{\gamma}^2 - 1\right)\left(\Theta_{\rm II}\cos^2\vartheta + \Theta_\perp \sin^2\vartheta\right)}$$

$$\times e^{-\frac{3}{\Theta_{\rm II}+2\Theta_\perp} \cdot \tilde{\gamma}} \left(\tilde{\gamma}^2 - 1\right)^{\frac{1}{2}} \tilde{\gamma} \, \mathrm{d}\tilde{\gamma} \mathrm{d}\cos\vartheta \, \mathrm{d}\phi,$$

where we have used $\Theta = \frac{1}{3}\left(2\Theta_\perp + \Theta_{\rm II}\right)$ (see Sect. 4.2). In Fig. 4 we plot the internal energy $\frac{1}{E_0} U(\Theta; A)$, where $\left(\Theta_\perp = \frac{3}{A+2}\Theta, \Theta_{\rm II} = \frac{3A}{A+2}\Theta\right)$ and $A \equiv \sqrt{2}\frac{\Theta_{\rm II}}{\Theta_\perp} = \frac{\Theta_z}{\Theta_x}$ defines the anisotropy. We observe that the internal energy is maximized in the isotropic case ($A=1$). Also, the internal energy becomes independent of the anisotropy only at the classical limit of $c \to \infty$.

4.2 Partition of temperature into its components

Finally, we show the partition of temperature T to its directional components T_i. This is shown by deriving the mean square of momentum:

$$\frac{\left\langle p^2 c^2 \right\rangle}{E_0^2} = \sum_{i=1}^{d} \frac{\left\langle p_i^2 c^2 \right\rangle}{E_0^2} = \sum_{i=1}^{d} \frac{\Theta_i}{\Theta} \frac{\left\langle \tilde{p}_i^2 c^2 \right\rangle}{E_0^2} \tag{46a}$$

$$= \sum_{i=1}^{d} \frac{\Theta_i}{\Theta} \frac{\left\langle \tilde{p}^2 c^2 \right\rangle}{E_0^2} \langle l_i \rangle = \frac{\left\langle \tilde{p}^2 c^2 \right\rangle}{E_0^2} \frac{\frac{1}{d}\sum_{i=1}^{d}\Theta_i}{\Theta} = \frac{\left\langle \tilde{p}^2 c^2 \right\rangle}{E_0^2},$$

where

$$\frac{\left\langle p^2 c^2 \right\rangle}{E_0^2} = \left\langle \gamma^2 - 1 \right\rangle = \int_1^{\infty} P_{\rm MJ}(\gamma\,;\Theta)(\gamma^2 - 1)\,\mathrm{d}\gamma \tag{46b}$$

$$= \int_1^{\infty} P_{\rm MJ}(\tilde{\gamma}\,;\Theta)(\tilde{\gamma}^2 - 1)\,\mathrm{d}\tilde{\gamma} = \left\langle \tilde{\gamma}^2 - 1 \right\rangle = \frac{\left\langle \tilde{p}^2 c^2 \right\rangle}{E_0^2}.$$

Note that $\langle l_i \rangle = \frac{1}{d}$. For example, for $d=3$ we have $\langle l_z \rangle = \int_{-1}^{1} \cos^2\vartheta \, \mathrm{d}\cos\vartheta = \frac{1}{3}$ and, thus, due to symmetry we obtain $\langle l_x \rangle = \langle l_y \rangle = \langle l_z \rangle = \frac{1}{3}$. Hence,

$$\Theta = \frac{1}{d}\sum_{i=1}^{d}\Theta_i \text{ or } T = \frac{1}{d}\sum_{i=1}^{d} T_i. \tag{47}$$

This shows the additivity of the temperature directional components characterizing each dimension, as required by the conditions in Sect. 3.1.

4.3 Entropy

The Boltzmann–Gibbs entropic formulation, which is aligned with the classical kinetic theory and the Maxwell distribution (Livadiotis and McComas, 2009), is better known for the discrete probability distribution $\{p_k\}_{k-1}^{W}$, that is,

$$S(\{p_k\}) = -\sum_{k=1}^{W} p_k \ln p_k. \tag{48a}$$

In the continuous description, however, this is given by (Livadiotis, 2014)

$$\int_{-\infty}^{\infty} \left[P_{\rm MJ}(\boldsymbol{p}\,;\Theta)\sigma^d\right] \frac{\mathrm{d}p_1 \ldots \mathrm{d}p_d}{\sigma^d} = 1, \tag{48b}$$

where the momentum scale σ is expressed in terms of the Planck constant h (and the plasma density n and temperature T or $\theta \equiv \sqrt{\frac{2k_{\rm B}T}{m}}$), derived within a semi-classical statistical/quantum mechanical approach (Livadiotis and McComas, 2013; Livadiotis, 2014):

$$\sigma \cong \left(\frac{n}{e}\right)^{\frac{1}{d}} h. \tag{49}$$

(Note that collisionless plasmas may be characterized by a different quantization constant, Livadiotis and McComas, 2013, 2014b; Livadiotis, 2016; Livadiotis and Desai, 2016.)

Having introduced the momentum scale σ, this is included in the expression of the normalization constant N, or the partition function $Z \equiv 1/N$; then, we have the distribution function as

$$P_{\mathrm{MJ}}(\boldsymbol{p};\boldsymbol{\Theta}) = N(\boldsymbol{\Theta}) \cdot \exp\left[-\frac{1}{\Theta}\cdot\sqrt{1+\sum_{i=1}^{d}\frac{\Theta}{\Theta_i}\left(\frac{p_i}{mc}\right)^2}\right],$$
$$N(\boldsymbol{\Theta}) \equiv \pi^{\frac{1-d}{2}} 2^{-\frac{d+1}{2}} \left(\frac{\sigma}{mc}\right)^d \times \Theta^{\frac{1}{2}} \prod_{i=1}^{d} \Theta_i^{-\frac{1}{2}} K_{\frac{d+1}{2}}\left(\frac{1}{\Theta}\right)^{-1}, \tag{50a}$$

the partition function as

$$Z = \int_{-\infty}^{\infty} \exp\left[-\frac{1}{\Theta}\cdot\sqrt{1+\sum_{i=1}^{d}\frac{\Theta}{\Theta_i}\left(\frac{p_i}{mc}\right)^2}\right] \frac{\mathrm{d}p_1\ldots\mathrm{d}p_d}{\sigma^d}$$
$$= \pi^{\frac{d-1}{2}} 2^{\frac{d+1}{2}} \left(\frac{\sigma}{mc}\right)^{-d} \cdot \Theta^{-\frac{1}{2}} \prod_{i=1}^{d} \Theta_i^{\frac{1}{2}} K_{\frac{d+1}{2}}\left(\frac{1}{\Theta}\right), \tag{50b}$$

the mean value of a momentum function $f(\boldsymbol{p})$ as

$$\langle f \rangle = \int_{-\infty}^{\infty} \left[P_{\mathrm{MJ}}(\boldsymbol{p};\boldsymbol{\Theta})\sigma^d\right] f(\boldsymbol{p}) \frac{\mathrm{d}p_1\ldots\mathrm{d}p_d}{\sigma^d} = 1, \tag{50c}$$

and the entropy as

$$\frac{1}{k_{\mathrm{B}}} S = -\int_{-\infty}^{\infty} \left[P_{\mathrm{MJ}}(\boldsymbol{p};\boldsymbol{\Theta})\sigma^d\right] \times \ln[P_{\mathrm{MJ}}(\boldsymbol{p};\boldsymbol{\Theta})\sigma^d] \frac{\mathrm{d}p_1\ldots\mathrm{d}p_d}{\sigma^d}. \tag{50d}$$

The entropy is analytically derived, below. First, we derive this for the isotropic case:

$$\frac{1}{k_{\mathrm{B}}} S = -\int_{-\infty}^{\infty} \left[P_{\mathrm{MJ}}(\boldsymbol{p};\Theta)\sigma^d\right] \cdot \ln\left[P_{\mathrm{MJ}}(\boldsymbol{p};\Theta)\sigma^d\right] \frac{\mathrm{d}p_1\ldots\mathrm{d}p_d}{\sigma^d}$$
$$= \int_{-\infty}^{\infty} P_{\mathrm{MJ}}(\boldsymbol{p};\Theta) \cdot \left[-\ln N + \gamma(\boldsymbol{p})/\Theta\right] \mathrm{d}p_1\ldots\mathrm{d}p_d$$
$$= -\ln N + \frac{1}{\Theta}\int_{-\infty}^{\infty} P_{\mathrm{MJ}}(\boldsymbol{p};\Theta) \cdot \gamma(\boldsymbol{p})\, \mathrm{d}p_1\ldots\mathrm{d}p_d$$
$$= -\ln N + N \cdot \left(\frac{\sigma}{mc}\right)^{-d} \frac{1}{\Theta} \int_{-\infty}^{\infty} e^{-\gamma(\boldsymbol{p})/\Theta} \gamma(\boldsymbol{p}) \frac{\mathrm{d}p_1\ldots\mathrm{d}p_d}{(mc)^d}$$
$$= -\ln N + N \cdot \left(\frac{\sigma}{mc}\right)^{-d} \frac{1}{2} B_d \frac{1}{\Theta} \int_{0}^{\infty} e^{-\frac{\gamma}{\Theta}} \gamma \frac{p^{d-1}\mathrm{d}p}{(mc)^d}$$
$$= -\ln N + N \cdot \left(\frac{\sigma}{mc}\right)^{-d} B_d \frac{1}{\Theta} \int_{1}^{\infty} e^{-\frac{\gamma}{\Theta}} \left(\gamma^2-1\right)^{\frac{d}{2}-1} \gamma^2 \mathrm{d}\gamma$$
$$= -\ln N + N \cdot \left(\frac{\sigma}{mc}\right)^{-d} B_d \frac{1}{\Theta} \cdot (I_d + I_{d-2})$$
$$= -\ln N + \left[d + \frac{\frac{1}{\Theta} K_{\frac{d-1}{2}}\left(\frac{1}{\Theta}\right)}{K_{\frac{d+1}{2}}\left(\frac{1}{\Theta}\right)}\right]$$
$$= \frac{U}{E_0\Theta} - \ln N.$$

Hence, we end up with

$$S = \frac{U}{T} - k_{\mathrm{B}} \ln N = \frac{U}{T} + k_{\mathrm{B}} \ln Z. \tag{51}$$

For the anisotropic case, we have

$$\frac{1}{k_{\mathrm{B}}} S = -\int_{-\infty}^{\infty} \left[P_{\mathrm{MJ}}(\boldsymbol{p};\boldsymbol{\Theta})\sigma^d\right] \cdot \ln\left[P_{\mathrm{MJ}}(\boldsymbol{p};\boldsymbol{\Theta})\sigma^d\right] \frac{\mathrm{d}p_1\ldots\mathrm{d}p_d}{\sigma^d} \tag{52}$$
$$= \int_{-\infty}^{\infty} P_{\mathrm{MJ}}(\boldsymbol{p};\boldsymbol{\Theta}) \cdot \left[-\ln N(\boldsymbol{\Theta}) + \gamma(\boldsymbol{p},\boldsymbol{\Theta})/\Theta\right] \mathrm{d}p_1\ldots\mathrm{d}p_d$$
$$= -\ln N(\boldsymbol{\Theta}) + \frac{1}{\Theta}\int_{-\infty}^{\infty} P_{\mathrm{MJ}}(\widetilde{\boldsymbol{p}};\Theta) \cdot \widetilde{\gamma}(\widetilde{\boldsymbol{p}}) \mathrm{d}\widetilde{p}_1\ldots\mathrm{d}\widetilde{p}_d,$$

because

$$P_{\mathrm{MJ}}(\boldsymbol{p};\boldsymbol{\Theta}) \cdot \gamma(\boldsymbol{p},\boldsymbol{\Theta})$$
$$= N(\boldsymbol{\Theta}) \cdot \exp\left[-\frac{1}{\Theta}\cdot\sqrt{1+\sum_{i=1}^{d}\frac{\Theta}{\Theta_i}\left(\frac{p_i}{mc}\right)^2}\right] \times \sqrt{1+\sum_{i=1}^{d}\frac{\Theta}{\Theta_i}\left(\frac{p_i}{mc}\right)^2}$$
$$= N(\Theta) \cdot \exp\left[-\frac{1}{\Theta}\cdot\sqrt{1+\sum_{i=1}^{d}\left(\frac{\widetilde{p}_i}{mc}\right)^2}\right]$$

$$\times\sqrt{1+\sum_{i=1}^{d}\left(\frac{\widetilde{p}_i}{mc}\right)^2}$$
$$= P_{\mathrm{MJ}}(\widetilde{\boldsymbol{p}};\Theta)\cdot\widetilde{\gamma}(\widetilde{\boldsymbol{p}}).$$

Then,

$$\frac{1}{k_\mathrm{B}}S = \frac{1}{k_\mathrm{B}}S_\mathrm{iso}+\ln\left[\frac{N(\Theta)}{N(\boldsymbol{\Theta})}\right] = \frac{1}{k_\mathrm{B}}S_\mathrm{iso}+\frac{1}{2}\sum_{i=1}^{d}\ln\left(\frac{\Theta_i}{\Theta}\right), \tag{53}$$

where S_iso is the entropy for the isotropic case, given by Eq. (51). In general, the product of elements of constant sum (Eq. 47) is maximized when all the elements are equal. Indeed, by maximizing the quantity $\sum_i\left[\ln\left(\frac{\Theta_i}{\Theta}\right)\right]$ under the constraint of constant $\sum_i\left(\frac{\Theta_i}{\Theta}\right)$ with a Lagrange coefficient $(\lambda-1)$, we have

$$0=\sum\nolimits_i\left[\ln\left(\frac{\Theta_i}{\Theta}\right)+(\lambda-1)\cdot\left(\frac{\Theta_i}{\Theta}\right)\right]$$
$$=\Theta_j^{-1}-[\lambda(d-1)+1]\,\Theta^{-1}=0$$
$$\Rightarrow\Theta_j=[\lambda(d-1)+1]^{-1}:\ \text{constant}$$

(Note that several thermodynamical properties can be found in Cercignani and Kremer, 2002.)

4.4 Thermodynamic relations

Here, we show four basic relations of thermodynamics using the isotropic relations. Up to now, there is no consistent thermodynamic theory for anisotropic temperature. Equation (51) gives the first thermodynamic relation, from which we easily derive the second, the free-energy relation:

$$S=U/T+k_\mathrm{B}\ln Z, \tag{54}$$
$$F\equiv U-TS=-k_\mathrm{B}T\cdot\ln Z.$$

We now proceed to the thermodynamic relation that defines temperature (Livadiotis and McComas, 2009):

$$\frac{1}{k_\mathrm{B}}\frac{\partial S}{\partial\Theta}=\frac{1}{\Theta}\frac{\partial\left(\frac{1}{E_0}U\right)}{\partial\Theta}-\frac{1}{\Theta^2}\frac{U}{E_0}+\frac{\partial\ln Z}{\partial\Theta}. \tag{55a}$$

From Eq. (14c), we derive $\frac{\partial\ln Z}{\partial\Theta}$:

$$\frac{\partial\ln Z}{\partial\Theta}=\frac{d-1}{2}\frac{1}{\Theta}-\frac{1}{\Theta^2}\frac{K'_{\frac{d+1}{2}}\left(\frac{1}{\Theta}\right)}{K_{\frac{d+1}{2}}\left(\frac{1}{\Theta}\right)} = \mathrm{d}\frac{1}{\Theta}+\frac{1}{\Theta^2}\frac{K_{\frac{d-1}{2}}\left(\frac{1}{\Theta}\right)}{K_{\frac{d+1}{2}}\left(\frac{1}{\Theta}\right)}=\frac{1}{\Theta^2}\frac{U}{E_0} \tag{55b}$$

because of Eq. (42c) and the recurrence relation,

$$\frac{K_{a-1}(x)}{K_a(x)}+\frac{K'_a(x)}{K_a(x)}=-\frac{a}{x}\ . \tag{56}$$

Hence, substituting Eq. (55b) into Eq. (55a), we obtain the third thermodynamic relation, the one that defines the temperature via the entropy (thermodynamic definition of temperature; Livadiotis and McComas, 2009; Livadiotis, 2015):

$$\frac{1}{k_\mathrm{B}}\frac{\partial S}{\partial\Theta}=\frac{1}{\Theta}\frac{\partial\left(\frac{1}{E_0}U\right)}{\partial\Theta}\Rightarrow\frac{\partial S}{\partial U}=\frac{k_\mathrm{B}}{E_0\Theta}=\frac{1}{T}. \tag{57}$$

This equation guarantees that the parameter Θ, selected to characterize the temperature (normalized to E_0) in MJ distribution, was correct. Finally, from the relations $U=U(\Theta)$ and $\ln Z=\ln Z(\Theta)$ in Eqs. (42c) and (55b), we find the fourth thermodynamic relation:

$$\Theta^2\frac{\partial\ln Z}{\partial\Theta}-\frac{U}{E_0} \tag{58}$$
$$=-\frac{d+1}{2}\Theta-\left[\frac{K_{\frac{d-1}{2}}\left(\frac{1}{\Theta}\right)}{K_{\frac{d+1}{2}}\left(\frac{1}{\Theta}\right)}+\frac{K'_{\frac{d+1}{2}}\left(\frac{1}{\Theta}\right)}{K_{\frac{d+1}{2}}\left(\frac{1}{\Theta}\right)}\right]=0$$
$$\Rightarrow\ \frac{U}{E_0}=\Theta^2\frac{\partial\ln Z}{\partial\Theta}=-\frac{\partial\ln Z}{\partial\left(\frac{1}{\Theta}\right)}\ \text{or}$$
$$U=-\frac{\partial\ln Z}{\partial(k_\mathrm{B}T)^{-1}}$$
$$=-\frac{\partial\ln Z}{\partial\beta_\mathrm{th}}\text{with }\beta_\mathrm{th}\equiv(k_\mathrm{B}T)^{-1}.$$

Note that the investigation to derive the thermodynamic definition of the temperature in the case of the anisotropic MJ, or even the classical MB, distributions is still in process.

5 Conclusions

The paper developed a model for the anisotropic Maxwell–Jüttner distribution, and examined its properties and thermodynamics. The Maxwell–Jüttner distribution is useful in space, geological, and other plasmas where high energy particles often reach the relativistic limits where the Maxwell–Boltzmann distribution is not applicable. On the other hand, these plasmas are known to be characterized by anisotropic Maxwell–Boltzmann distributions as long as particle energies are non-relativistic; thus, we have no reason to expect isotropic distributions at relativistic high energies. Therefore, it is necessary to deduce a consistent formulation of anisotropic Maxwell–Jüttner distributions.

First, it provided the characteristic conditions that a consistent and well-defined anisotropic model of Maxwell–Jüttner distributions needs to fulfill. Then, guided by these conditions, the paper derived a consistent model, and examined its properties and thermodynamics.

In particular, the paper showed and discussed the following analytical developments:

– provided the conditions for modeling consistent anisotropic Maxwell–Jüttner distributions,

– examined several models, showing their possible advantages and failures,

– derived a consistent anisotropic model that fulfills all the desired characteristic conditions. For example, the correspondence with both the classical and isotropic limits, for $c \to \infty$ (anisotropic MB) and for $\frac{T_i}{T} \to 1$ (isotropic MJ), is fulfilled; and

– studied the properties and thermodynamics of this model, e.g., the internal energy, the partition of temperature to its components, and the entropy. The anisotropic internal energy and entropy are both depended on the anisotropy; they are maximized for the isotropic case, while both become independent of the anisotropy at the classical case of $c \to \infty$.

It is now straightforward for space physics researchers to use the derived analytical model in applications. The next goals may be to (i) show the connection with thermodynamics for the anisotropic model, (ii) derive other different anisotropic model(s), and (iii) to establish both the isotropic/anisotropic models within the framework of kappa distributions for particle systems described by stationary states out of thermal equilibrium (e.g., Livadiotis, 2015).

The topical editor, E. Roussos, thanks two anonymous referees for help in evaluating this paper.

References

Abramowitz, M. and Stegun, I. A. (Eds.): Handbook of mathematical functions with formulas, graphs, and mathematical tables, 9th Edn., Dover, New York, 376 pp., 1972.

Bavassano Cattaneo, M. B., Marcucci, M. F., Retinò, A., Pallocchia, G., Rème, H., Dandouras, I., Kistler, L. M., Klecker, B., Carlson, C. W., Korth, A., McCarthy, M., Lundin, R., and Balogh, A.: Kinetic signatures during a quasi-continuous lobe reconnection event: Cluster Ion Spectrometer observations, J. Geophys. Res., 111, A09212, doi:10.1029/2006JA011623, 2006

Cai, C. L., Dandouras, I., Rème, H., Cao, J. B., Zhou, G. C., and Parks, G. K.: Cluster observations on the thin current sheet in the magnetotail, Ann. Geophys., 26, 929–940, doi:10.5194/angeo-26-929-2008, 2008.

Cercignani, C. and Kremer, G. M.: The relativistic Boltzmann equation: theory and applications, Birkhäuser Verlag, Basel, Switzerland, 2002.

Dunkel, J., Talkner, P., and Hänggi, P.: Relative entropy, Haar measures and relativistic canonical velocity distributions, New J. Phys., 9, 144 pp., doi:10.1088/1367-2630/9/5/144, 2007.

Feldman, W., Asbridge, J. R., Bame, S. J., Montgomery, M. D., and Gary, S. P.: Solar wind electrons, J. Geophys. Res., 80, 4181–4196, 10.1029/JA080i031p04181, 1975.

Gary, S. P.: The mirror and ion cyclotron anisotropy instabilities, J. Geophys. Res., 97, 8519, doi:10.1029/92JA00299, 1992.

Horwitz, L. P., Schieve, W. C., and Piron, C.: Gibbs ensembles in relativistic classical and quantum mechanics, Ann. Phys.-New York, 137, 306–340, doi:10.1016/0003-4916(81)90199-8, 1981.

Jüttner, F.: Das Maxwellsche Gesetz der Geschwindigkeitsverteilung in der Relativtheorie, Ann. Phys.-Berlin, 339, 856–882, doi:10.1002/andp.19113390503, 1911.

Kasper, J. C., Lazarus, A. J., and Gary, S. P.: Wind/SWE observations of firehose constraint on solar wind proton temperature anisotropy, Geophys. Res. Lett., 29, 1839, doi:10.1029/2002GL015128, 2002.

Krall, N. A. and Trivelpiece, A. W.: Principles of Plasma Physics, McGraw-Hill: Kogakusha, 1973.

Lehmann, E.: Covariant equilibrium statistical mechanics, J. Math. Phys., 47, 023303, doi:10.1063/1.2165771, 2006.

Livadiotis, G.: Approach to general methods for fitting and their sensitivity, Physica A, 375, 518–536, doi:10.1016/j.physa.2006.09.027, 2007.

Livadiotis, G.: Lagrangian temperature: Derivation and physical meaning for systems described by kappa distributions, Entropy, 16, 4290–4308, doi:10.3390/e16084290, 2014.

Livadiotis, G.: Statistical background and properties of kappa distributions in space plasmas, J. Geophys. Res., 120, 1607–1619, doi:10.1002/2014JA020825, 2015.

Livadiotis, G.: Evidence for a large-scale Compton wavelength in space plasmas, Astrophys. J. Suppl. Ser., 223, 13 pp, 2016.

Livadiotis, G. and Desai, M. I.: Plasma-field coupling at small length scales in solar wind near 1 au, Astrophys. J., 829, 88, 14 pp, doi:10.3847/0004-637X/829/2/88, 2016.

Livadiotis, G. and McComas, D. J.: Beyond kappa distributions: Exploiting Tsallis statistical mechanics in space plasmas, J. Geophys. Res., 114, A11105, doi:10.1029/2009JA014352, 2009.

Livadiotis, G. and McComas, D. J.: Evidence of large scale phase space quantization in plasmas, Entropy, 15, 1118–1134, doi:10.3390/e15031118, 2013.

Livadiotis, G. and McComas, D. J.: Electrostatic shielding in plasmas and the physical meaning of the Debye length, J. Plasma Phys., 80, 341–378, doi:10.1017/S0022377813001335, 2014a.

Livadiotis, G. and McComas, D. J.: Large-Scale phase-space quantization from local correlations in space plasmas, J. Geophys. Res., 119, 3247–3258, doi:10.1002/2013JA019622, 2014b.

Matteini, L., Landi, S., Hellinger, P., Pantellini, F., Maksimovic, M., Velli, M., Goldstein, B. E., ans Marsch, E.: Evolution of the solar wind proton temperature anisotropy from 0.3 to 2.5 AU, Geophys. Res. L., 34, L20105, doi:10.1029/2007GL030920, 2007.

Nishino, M. N., Fujimoto, M., Terasawa, T., Ueno, G., Maezawa, K., Mukai, T., and Saito, Y.: Geotail observations of temperature anisotropy of the two-component protons in the dusk plasma sheet, Ann. Geophys., 25, 769–777, doi:10.5194/angeo-25-769-2007, 2007.

Olsen, S. P. and Leer, E.: A study of solar wind acceleration based on gyrotropic transport equations, J. Geophys. Res., 104, 9963–9973, doi:10.1029/1998JA900152, 1999.

Phillips, J. L. and Gosling, J. T.: Radial evolution of solar wind thermal electron distributions due to expansion and collisions, J. Geophys. Res., 95, 4217–4228, doi:10.1029/JA095iA04p04217, 1990.

Pilipp, W. and Morfill, G.: The plasma mantle as the origin of the plasma sheet, magnetospheric particles and fields astrophysics and space science library, 58, 55–66, doi:10.1007/978-94-010-1503-5_6, 1976.
Pilipp, W. G., Muehlhaeuser, K.-H., Miggenrieder, H., Montgomery, M. D., and Rosenbauer, H.: Characteristics of electron velocity distribution functions in the solar wind derived from the Helios plasma experiment, J. Geophys. Res., 92, 1075–1092, doi:10.1029/JA092iA02p01075, 1987.
Renuka, G. and Viswanathan, K. S.: Instabilities of the whistler mode in the magnetosphere, Indian J. Radio Space Phys., 7, 248–253, 1978.
Rezzola, L. and Zanotti, O., Relativistic hydrodynamics, Oxford University Press, Oxford, UK, 2013, 99 pp., 2013
Sckopke, N., Paschmann, G., Brinca, A. L., Carlson, C. W., and Luhr, H.: Ion thermalization in quasi-perpendicular shocks involving reflected ions, J. Geophys. Res., 95, 6337, doi:10.1029/JA095iA05p06337, 1990.
Štverák, S., Trávníček, P., Maksimovic, M., Marsch, E., Fazakerley, A. N., and Scime, E. E.: J. Geophys. Res., 113, A03103, doi:10.1029/2007JA012733, 2008.
Treumann, R. A. and Baumjohann, W.: Anisotropic Jüttner (relativistic Boltzmann) distribution, Ann. Geophys., 34, 737–738, doi:10.5194/angeo-34-737-2016, 2016.
Tsurutani, B. T., Smith, E. J., Anderson, R. R., Ogilvie, K. W., Scudder, J. D., Baker, D. N., and Bame, S. J.: Lion roars and Non-oscillatory drift mirror waves in the magnetosheath, J. Geophys. Res., 87, 6060, doi:10.1029/JA087iA08p06060, 1982.
Winglee, R. and Harnett, E.: The influence of temperature anisotropies in controlling the development of magnetospheric substorms, arXiv:1605.01399, 2016.