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Ionospheric and boundary contributions to the Dessler-Parker-Sckopke formula for Dst

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Abstract. The Dessler-Parker-Sckopke formula for the disturbance magnetic field averaged over the Earth's surface, universally used to interpret the geomagnetic Dst index, can be generalized, by using the well known method of deriving it from the virial theorem, to include the effects of ionospheric currents. There is an added term proportional to the global integral of the vertical mechanical force that balances the vertical component of the Lorentz force $\boldsymbol{J}\times\boldsymbol{B}/c$ in the ionosphere; a downward mechanical force reduces, and an upward increases, the depression of the magnetic field. If the vertical component of the ionospheric Ohm's law holds exactly, the relevant force on the plasma is the collisional friction between the neutral atmosphere and the vertically flowing plasma. An equal and opposite force is exerted on the neutral atmosphere and thus appears in *its* virial theorem. The ionospheric effect on Dst can then be related to the changes of kinetic and gravitational energy contents of the neutral atmosphere; since these changes are brought about by energy input from the magnetosphere, there is an implied upper limit to the effect on Dst which in general is relatively small in comparison to the contribution of the plasma energy content in the magnetosphere. Hence the Dessler-Parker-Sckopke formula can be applied without major modification, even in the case of strong partial ring currents; the ionospheric closure currents implied by the local time asymmetry have only a relatively small effect on the globally averaged disturbance field, comparable to other sources of uncertainty. When derived from the virial theorem applied to a bounded volume (e.g. the magnetosphere bounded by the magnetopause and a cross-section of the magnetotail), the Dessler-Parker-Sckopke formula contains also several boundary surface terms which can be identified as contributions of the magnetopause (Chapman-Ferraro) and of the magnetotail currents.

Keywords. Magnetospheric physics (Magnetosphere-ionosphere interactions; Solar wind-magnetosphere interactions; Storms and substorms)

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1 Introduction

The Dessler-Parker-Sckopke theorem gives the magnetic field perturbation at the Earth due to plasma trapped in the terrestrial dipole magnetic field and is therefore of fundamental importance in relating the geomagnetic Dst index to the properties of plasma in the magnetosphere. Written in modern notation, their formula is

$$\boldsymbol{\mu}\cdot\boldsymbol{b}(0)=2U_K \qquad (1)$$

where $\boldsymbol{\mu}$ is the dipole moment, $\boldsymbol{b}(0)$ is the magnetic disturbance field, nominally evaluated at the center of the Earth but also equal to the (vector) average over the surface of the globe, and U_K is the total kinetic energy content of plasma in the magnetosphere. Equation (1) was derived (Dessler and Parker, 1959) by summing the currents associated with the gradient, curvature, and magnetization drifts of plasma particles trapped in a dipole magnetic field (assuming linearity and axial symmetry: drifts in an unperturbed field, uniform distribution in longitude), at first for two special distributions (isotropic and flat) of pitch angles and later (Sckopke, 1966) for arbitrary pitch angle distributions. Subsequently several groups (Maguire and Carovillano, 1967; Baker and Hurley, 1967; Carovillano and Maguire, 1968; Olbert et al., 1968) gave a simpler, rigorous derivation by an argument related to the virial theorem, without the restrictive assumptions of linearity and axial symmetry, generalizing the formula to

$$\boldsymbol{\mu}\cdot\boldsymbol{b}(0)=2U_K+U_b \qquad (2)$$

where U_b is the total energy content in the disturbance magnetic field:

$$U_b=\int d\boldsymbol{r}\; b^2/8\pi \qquad \boldsymbol{b}\equiv\boldsymbol{B}-\boldsymbol{B}_{dipole} \qquad (3)$$

(Gaussian units are used throughout this paper). The virial theorem method can be further generalized to take into account the boundary of the magnetosphere (Siscoe, 1970), adding boundary surface terms to Eq. (2) that represent disturbance fields arising from the boundary regions (ignored in

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earlier derivations that effectively treated the magnetosphere as extending to infinity). The entire development has been definitively reviewed by Carovillano and Siscoe (1973).

Although the Dessler-Parker-Sckopke theorem might thus seem to be a closed chapter at least as far as theory is concerned, there still are some outstanding issues which are addressed in this paper:

1. First and foremost, there remains at least one fundamental problem: none of the existing derivations known to me have included ionospheric currents, which are potentially important particularly for configurations with strong local-time asymmetries (closure of "partial ring currents") and have sometimes (e.g. Liemohn et al., 2001; Liemohn, 2003) been adduced as a reason for questioning the applicability of the theorem. What *has* been done by many authors (e.g. Parker, 1966; Siscoe and Crooker, 1974; Crooker and Siscoe, 1974, 1981; Friedrich et al., 1999; Munsani, 2000, and others) is to calculate, on the basis of some specific model, the magnetic effects of a complete asymmetric current system (e.g., partial ring current or substorm current wedge), including the ionospheric closure together with all the other currents. However useful such model results may be in their particular context, it is not clear how they are to be combined with the Dessler-Parker-Sckopke formula; for instance, should the kinetic energy of plasma in the partial ring current be included in the U_K term of Eqs. (1) and (2)?

2. The entire theoretical development beyond the original papers of Dessler and Parker (1959) and Sckopke (1966) still seems to be little known and poorly understood. The boundary surface terms in particular are almost universally ignored (or sometimes misunderstood; see discussion in Appendix A), boundary effects on Dst being taken into account by ad hoc "corrections" instead. On a more basic level, it is hardly ever appreciated that the Dessler-Parker-Sckopke theorem, although formally equivalent to the Biot-Savart law (Vasyliūnas, 2001) and of course originally derived by applying it, has a physical basis quite distinct from (and more restrictive than) the Biot-Savart law.

In this paper I present, in Sect. 2, a rigorous and general derivation of the Dessler-Parker-Sckopke formula from the virial theorem, taking into account ionospheric effects, bounded volume, and also time-derivative terms (neglected from the start in most discussions). In Sect. 3, I evaluate the ionospheric contributions and show that they are relatively small and can for most purposes be neglected in practice. In Sect. 4, I consider the boundary surface terms and their relation to the usual, empirically derived, pressure corrections and magnetotail current contributions. The end result (Sect. 5) is a version of the Dessler-Parker-Sckopke expression for Dst that is formally complete and contains all the relevant terms, with each term unambiguously defined and all terms derived, in a uniform consistent fashion, from clearly identified physical premises.

Two important aspects, concerning applications of the Dessler-Parker-Sckopke theorem rather than the theorem itself, are beyond the scope of this paper and will not be dealt with. First, the conversion between the nominal disturbance field $\boldsymbol{b}(0)$ that appears in the theorem and the Dst index that is determined from observations (e.g., Chapter 8 of Mayaud, 1980, and references therein) is here treated as given entirely by the conventionally used scaling factor, with at most passing mention of the implied assumptions and outstanding problems. Second, the use of the theorem for studying magnetospheric and geomagnetic-storm dynamics, in particular for predicting Dst time series on the basis of solar-wind and other inputs, is a separate topic and constitutes an extensive field of research in its own right (e.g. Burton et al., 1975; O'Brien and McPherron, 2000, 2002; McPherron and O'Brien, 2001; Liemohn et al., 2001, 2002; Jordanova et al., 1998, 2003; Siscoe et al., 2005a,b, and many others); elsewhere (Vasyliūnas, 2006) I discuss how the magnetotail term derived here affects some prediction equations for Dst.

2 Derivation from the virial theorem

The virial theorem in general uses the momentum equation (taking its dot product with the radius vector and integrating by parts) to infer global constraints on the various energies within the system under consideration. (Simplest and best-known example: in a stable self-gravitating system, potential energy plus twice kinetic energy equals zero.) In the particular application that leads to the Dessler-Parker-Sckopke theorem, the momentum equation that now includes the Lorentz force $\boldsymbol{J}\times\boldsymbol{B}/c$ is subjected to the procedure described above, followed by special manipulation in order to isolate the magnetic disturbance field in the region near the magnetic dipole. The mathematical derivation is presented in detail below, but the following physical point may be noted at the outset. The essential physical basis of the Dessler-Parker-Sckopke theorem is the assumed validity of the momentum equation; when the theorem is derived from the Biot-Savart law, the currents that appear in the Biot-Savart integral are those determined by the requirement that $\boldsymbol{J}\times\boldsymbol{B}/c$ balance the rest of the momentum equation. It is in this sense that, as noted in the introduction, the Dessler-Parker-Sckopke theorem is more restrictive than the Biot-Savart law. (This is also the reason why combining terms from the Dessler-Parker-Sckopke formula with disturbance fields calculated from model currents not constrained by the momentum equation is always problematic.)

2.1 Formal derivation

The starting point for the virial theorem is the momentum equation of the plasma

$$(\partial/\partial t)\,(\rho\boldsymbol{V}) + \nabla\cdot\mathbf{T} = \rho\boldsymbol{g} + \boldsymbol{f} \quad (4)$$

where $\mathbf{T}$ is the total stress tensor

$$\mathbf{T} = \rho\boldsymbol{V}\boldsymbol{V} + \mathbf{P} + \mathbf{I}\left(B^2/8\pi\right) - \boldsymbol{B}\boldsymbol{B}/4\pi \quad , \quad (5)$$

electric field terms have been neglected (assumption of quasi-neutrality and ${V_A}^2 \ll c^2$), gravity is for our purposes more conveniently included as the term $\rho \boldsymbol{g}$ on the right-hand side, although it could be added to $\mathbf{T}$ instead (see, e.g., Siscoe, 1970), and $\boldsymbol{f}$ represents all other forces not included in $\mathbf{T}$, in particular plasma-neutral collisions important in the ionosphere. (For a discussion of the virial theorem in a more general gravitational-hydromagnetic context, see Chandrasekhar, 1961, Chapter XIII, section 117).

Consider a volume bounded by a closed inner surface and a closed outer surface; the boundaries are to be precisely defined later but can be thought of as, roughly, the Earth and the magnetopause, respectively. Taking the dot product of Eq. (4) with the radius vector $\boldsymbol{r}$, integrating over the volume under consideration, and using the identity

$$(\nabla \cdot \mathbf{T}) \cdot \boldsymbol{r} = \nabla \cdot (\mathbf{T} \cdot \boldsymbol{r}) - \mathrm{Trace}\,(\mathbf{T}) \tag{6}$$

together with the relation between the trace of a stress tensor and the corresponding energy density yields

$$(d/dt) \int d\boldsymbol{r} \rho \boldsymbol{V} \cdot \boldsymbol{r} + \int d\boldsymbol{S} \cdot \mathbf{T} \cdot \boldsymbol{r} = 2U_K + U_B + U_G + \int d\boldsymbol{r}\; \boldsymbol{f} \cdot \boldsymbol{r} \tag{7}$$

where the surface integrals are over the boundaries of the volume; U_K and U_B represent the volume integrals of the kinetic energy density (including bulk flow and thermal energies) of the plasma and the energy density of the (total) magnetic field, respectively:

$$U_K = \int d\boldsymbol{r}\;\; (1/2) \left(\rho V^2 + \mathrm{Trace}\,(\mathbf{P})\right)$$
$$U_B = \int d\boldsymbol{r}\;\; B^2/8\pi \;\; . \tag{8}$$

The gravitational term U_G represents the integral

$$U_G = \int d\boldsymbol{r}\; \rho \boldsymbol{g} \cdot \boldsymbol{r} \tag{9}$$

which may in the present case, with the Earth excluded from the volume of integration and with $\boldsymbol{g}$ approximated as the spherically symmetric gravity field of the Earth (neglecting any non-symmetric terms as well as self-gravity of the mass within the volume), be viewed as the gravitational energy; note that it differs from the gravitational terms in the formulations of Chandrasekhar (1961) or Siscoe (1970), where these approximations are not made. The term U_G is always negative and may be treated as simply a reduction of U_K by an amount that, in the case of plasma, is easily shown to be negligible.

The first term on the left-hand side of Eq. (7) can be reformulated by invoking the relations

$$\begin{aligned}\rho \boldsymbol{V} \cdot \boldsymbol{r} &= \rho \boldsymbol{V} \cdot \nabla r^2/2 \\ &= \nabla \cdot \left(\rho \boldsymbol{V} r^2/2\right) - \left(r^2/2\right) \nabla \cdot \rho \boldsymbol{V} \\ &= \nabla \cdot \left(\rho \boldsymbol{V} r^2/2\right) + (\partial/\partial t)\, \rho r^2/2 \end{aligned} \tag{10}$$

where the last line follows from mass continuity; Eq. (7) can then be rewritten as

$$\frac{1}{2}\left(d^2/dt^2\right) \int d\boldsymbol{r}\; \rho r^2 + \int d\boldsymbol{S} \cdot \left[(\partial \rho \boldsymbol{V}/\partial t)\, r^2/2 + \mathbf{T} \cdot \boldsymbol{r}\right] = 2U_K + U_B + U_G + \int d\boldsymbol{r}\; \boldsymbol{f} \cdot \boldsymbol{r} \tag{11}$$

(cf. Rossi and Olbert, 1970, Problem 10.1). The integral in the first term on the left-hand side of Eq. (11) looks like (and is often described as) the moment of inertia of the system, which is not quite accurate because the radius vector in it is measured from an arbitrary origin rather than from the center of mass. By introducing, however, the center of mass $\boldsymbol{R}$ through the definitions

$$M\boldsymbol{R} \equiv \int d\boldsymbol{r}\; \rho\, \boldsymbol{r} \qquad M \equiv \int d\boldsymbol{r}\; \rho \tag{12}$$

it is possible to further rewrite Eq. (11), after extensive manipulation, as

$$\frac{1}{2} d^2\mathcal{I}/dt^2 + \int d\boldsymbol{S} \cdot \left[(\partial \rho \boldsymbol{V}/\partial t) \left|\boldsymbol{r} - \boldsymbol{R}\right|^2/2 + \mathbf{T} \cdot (\boldsymbol{r} - \boldsymbol{R})\right] = 2U_K - M\left|\dot{\boldsymbol{R}}\right|^2 + U_B + U_G^* + \int d\boldsymbol{r}\; \boldsymbol{f} \cdot (\boldsymbol{r} - \boldsymbol{R}) \tag{13}$$

where

$$U_G^* = \int d\boldsymbol{r}\; \rho \boldsymbol{g} \cdot (\boldsymbol{r} - \boldsymbol{R}) \tag{14}$$

and

$$\mathcal{I} \equiv \int d\boldsymbol{r}\; \rho \left|\boldsymbol{r} - \boldsymbol{R}\right|^2 = \int d\boldsymbol{r}\; \rho \left(r^2 - R^2\right) \tag{15}$$

is the properly defined moment of inertia; everything is now referred to the center of mass as the origin, and the kinetic energy of motion of the system as a whole has been subtracted.

Elegant though it may be, Eq. (13) as distinct from Eq. (11) is important only for systems subject to external forces sufficiently strong and asymmetric to produce appreciable acceleration of the system as a whole. For the magnetosphere anchored to the massive Earth this is certainly not the case, and throughout this paper I shall use Eq. (11) with $\boldsymbol{r}$ measured from the center of the Earth.

The essential application of the virial theorem is to systems that endure, that do not undergo either fast dispersion or fast collapse. The first term on the left-hand side of Eq. (11) or (13) (the second time derivative of the moment of inertia) can then be neglected, and balance of the remaining terms is the condition for the system to endure. Equivalently, if e.g. the (positive) energies U_K and U_B were not balanced by suitable negative terms, the system would disperse on a short time scale (of order Alfvén wave or typical particle crossing time). The (d^2/dt^2) term is dropped in the rest of this paper; this, however, does not necessarily imply neglect of any other time derivatives.

2.2 Steps toward the Dessler-Parker-Sckopke relation

To obtain the Dessler-Parker-Sckopke relation from the virial theorem, two steps are needed. First, choose as the lower boundary of the volume of integration the surface of a small Earth-centered sphere (of radius $R_s \sim R_E$), at which only the magnetic terms in the stress tensor are assumed significant. (For the later purpose of estimating the ionospheric contributions, it will be important that the surface lie below the ionosphere.) Evaluating the surface term at the lower boundary then transforms Eq. (11) into

$$\begin{aligned} -R_s{}^3 \int d\Omega \left[B^2/8\pi - \left(\boldsymbol{B}\cdot\hat{\boldsymbol{r}}\right)^2/4\pi\right] \\ = 2U_K + U_B + U_G \\ - \int d\boldsymbol{S}\cdot\left[(\partial\rho\boldsymbol{V}/\partial t)\,r^2/2 + \mathbf{T}\cdot\boldsymbol{r}\right] + \int d\boldsymbol{r}\;\boldsymbol{f}\cdot\boldsymbol{r} \end{aligned} \tag{16}$$

where the surface integral on the right-hand side refers only to the upper boundary, the integral $\int d\Omega$ over the lower boundary having been transferred to the left-hand side. Second, replace the momentum Eq. (4) by the corresponding equation containing a curl-free magnetic field $\boldsymbol{B}_d$ alone and no plasma,

$$\nabla\cdot\left[\mathbf{I}\left(B_d{}^2/8\pi\right) - \boldsymbol{B}_d\boldsymbol{B}_d/4\pi\right] = 0 \ , \tag{17}$$

and carry out the same procedure (dot product with $\boldsymbol{r}$, integration over the assumed volume) to obtain the counterpart of Eq. (16):

$$\begin{aligned} -R_s{}^3 \int d\Omega\left[B_d{}^2/8\pi - \left(\boldsymbol{B}_d\cdot\hat{\boldsymbol{r}}\right)^2/4\pi\right] \\ = U_{Bd} - \int d\boldsymbol{S}\cdot\left[\boldsymbol{r}\left(B_d{}^2/8\pi\right) - \left(\boldsymbol{B}_d\boldsymbol{B}_d/4\pi\right)\cdot\boldsymbol{r}\right] \end{aligned} \tag{18}$$

where

$$U_{Bd} \equiv \int d\boldsymbol{r}\; B_d{}^2/8\pi \tag{19}$$

Although $\boldsymbol{B}_d$, as its designation suggests, will soon be identified as the dipole field, the only property of $\boldsymbol{B}_d$ invoked so far is that, within the volume under consideration, it has zero curl and hence can be written as the gradient of a scalar potential, $\boldsymbol{B}_d = -\nabla\psi$.

Now subtract Eq. (18) from Eq. (16). The difference of the magnetic energies can be written as

$$\begin{aligned} U_B - U_{Bd} &= U_b + \int d\boldsymbol{r}\;\boldsymbol{b}\cdot\boldsymbol{B}_d/4\pi \\ &= U_b - \int d\boldsymbol{r}\,\boldsymbol{b}\cdot\nabla\psi/4\pi \\ &= U_b - \int d\boldsymbol{S}\cdot\,\boldsymbol{b}\,\psi/4\pi + R_s{}^3\int d\Omega\,\boldsymbol{b}\cdot\hat{\boldsymbol{r}}\;\psi/4\pi \end{aligned} \tag{20}$$

where U_b and $\boldsymbol{b}$ have been defined in expression (3). The third term in the last line of Eq. (20) can be transferred to the left-hand side of the difference equation, Eq. (16) minus Eq. (18), which then becomes

$$\begin{aligned} -R_s{}^3\int d\Omega\left\{\boldsymbol{b}\cdot\left[\boldsymbol{B}_d - \hat{\boldsymbol{r}}\left(2\boldsymbol{B}_d\cdot\hat{\boldsymbol{r}} - \psi/R_s\right)\right]/4\pi \right. \\ \left. + \left[b^2 - 2\left(\boldsymbol{b}\cdot\hat{\boldsymbol{r}}\right)^2\right]/8\pi\right\} . \end{aligned} \tag{21}$$

Assume now that $\boldsymbol{B}_d$ is indeed the dipole field. Then

$$\boldsymbol{B}_d\cdot\boldsymbol{r} = 2\psi = 2\boldsymbol{\mu}\cdot\hat{\boldsymbol{r}}/r^2 \tag{22}$$

and the quantity in [ ] in the first line of expression (21) can be rewritten as

$$-\nabla\psi - 3\psi\left(\hat{\boldsymbol{r}}/r\right) = -r^{-3}\nabla\left(r^3\psi\right) = -\boldsymbol{\mu}/r^3 \tag{23}$$

evaluated at $r=R_s$; the second line of expression (21) can be neglected to order b/B_d. Finally, the left-hand side of the difference equation, Eq. (16) minus Eq. (18), becomes

$$\int (d\Omega/4\pi)\;\boldsymbol{\mu}\cdot\boldsymbol{b} = \boldsymbol{\mu}\cdot\boldsymbol{b}(0)\ . \tag{24}$$

2.3 Expression for the disturbance field

With all this taken into account, and with the assumption of a gyrotropic pressure tensor, Eq. (16) minus Eq. (18) can be written, after some manipulation, as

$$\begin{aligned} \boldsymbol{\mu}\cdot\boldsymbol{b}(0) = 2U_K + U_b + U_G + \int d\boldsymbol{r}\;\boldsymbol{f}\cdot\boldsymbol{r} \\ - \int d\boldsymbol{S}\cdot\boldsymbol{r}\left[P_\perp + \left(B^2/8\pi\right)\right] \\ + \int d\boldsymbol{S}\cdot\left[\boldsymbol{B}_d\times\left(\boldsymbol{r}\times\boldsymbol{B}_d\right)\right]/8\pi \\ + \int d\boldsymbol{S}\cdot\boldsymbol{B}\left[\chi\boldsymbol{B} - \left(\boldsymbol{B}_d/2\right)\right]\cdot\boldsymbol{r}/4\pi \\ - \int d\boldsymbol{S}\cdot\left[(\partial\rho\boldsymbol{V}/\partial t)\,r^2/2 + \rho\boldsymbol{V}\left(\boldsymbol{V}\cdot\boldsymbol{r}\right)\right] \end{aligned} \tag{25}$$

where the factor $\chi \equiv 1 - \left[4\pi\left(P_\parallel - P_\perp\right)/B^2\right]$ incorporates any effects of pressure anisotropy. This is the complete formula, derived from the virial theorem, for the magnetic disturbance field at the center of the Earth. On the right-hand side, the first term is the classical Dessler-Parker-Sckopke expresion and the second its nonlinear generalization (Carovillano and Siscoe, 1973, and references therein); the third term is the (negative) gravitational effect, and the fourth is the ionospheric contribution, to be estimated in the next section. The remaining terms are surface integrals over the outer boundary (normal direction taken as pointing outward): the second line is the contribution of total pressure, previously derived by Siscoe (1970), the third line is a correction term to the second as discussed below, the fourth line is related to magnetic fields normal to the boundary (open magnetosphere), and the fifth (probably not important in practice, at least as long as the outer boundary lies within the magnetosphere) is related to bulk flow through the boundary. Note the signs of the various terms; $\boldsymbol{\mu}\cdot\boldsymbol{b}(0)>0$ corresponds to $Dst<0$.

The presence of the purely dipole-field expression in the third line and of the subtracted dipole field in the second term of the fourth line might seem somewhat peculiar. A formal check, however, on the correctness of Eq. (25) can be made by setting all the plasma terms to zero and $\boldsymbol{B}$ to the dipole field everywhere within the volume; then the only non-zero terms are the three surface integrals in lines 2, 3, and 4 (with $\boldsymbol{B}=\boldsymbol{B}_d$, $P_\perp=0$, and $\chi=1$), and they must add up to zero – as they indeed do, but only thanks to these "peculiar" terms! The expression in the third line can be evaluated explicitly (by redefining the normal direction as pointing inward and invoking the curl-free property of $\boldsymbol{B}_d$) and shown to equal the energy density of the dipole field integrated over the volume *external* to the outer boundary. The significance of this becomes apparent upon comparing Eq. (25) with the corresponding earlier result of Siscoe (1970, his Eq.(23)):

$$\boldsymbol{\mu}\cdot\boldsymbol{b}(0) = 2U_K + U_b{}^* - \int d\boldsymbol{S}\cdot\boldsymbol{r}\left[P_\perp + \left(B^2/8\pi\right)\right] \tag{26}$$

in our notation. (This equation was derived also from the virial theorem, by a slightly different but equivalent method: Siscoe (1970) takes the virial of a volume that includes the Earth and then subtracts the virial of the Earth, while I take from the start a volume that excludes the Earth.) Lines 4 and 5 of Eq. (25) are absent because Siscoe (1970) explicitly assumed the magnetosphere to be closed, with no field or flow across the magnetopause, and also he did not consider any ionospheric effects. What still needs to be explained is the absence in Eq. (26) of line 3 of Eq. (25). The reason lies in the different definitions of the energy of the disturbance field: U_b in Eq. (25) is the integral over the volume between the inner and the outer surface, while $U_b{}^*$ in Eq. (26) is the integral over all space. The two thus differ by the energy of the disturbance field integrated over the volume beyond the outer surface; for the closed magnetosphere assumed by Siscoe (1970), the disturbance field beyond the magnetopause is equal to minus the dipole field.

3 Ionospheric contribution

The ionospheric contribution to the dipole-aligned component of the disturbance field at the center of the Earth, given by the term $\int d\boldsymbol{r}\;\boldsymbol{f}\cdot\boldsymbol{r}$ in Eqs. (7), (11), (16), or (25), is proportional to the integral, over the entire ionosphere, of radial force density times radial distance. This dependence can be easily understood by noting that the Biot-Savart integral over the current distribution for the z component of the field at the origin can be written (cf. Eq. (A7) of Vasyliūnas, 2001) as

$$b_z(0) = \int d\boldsymbol{r}\;\left(\sin\theta/r^2\right)\hat{\boldsymbol{\phi}}\cdot\boldsymbol{J}/c \tag{27}$$

and that $\mu\sin\theta/r^2$ is simply rB_θ of the dipole field; multiplied by the dipole moment, the integrand is thus $\boldsymbol{r}\cdot\boldsymbol{J}\times\boldsymbol{B}/c$, the radial component of the Lorentz force which must be balanced by the sum of all the non-magnetic forces. The pressure gradient and flow acceleration forces do not appear explicitly, nor does gravity; they have been included already as part of the total energy integral U_K and the gravitational term U_G, respectively. The remaining "other force" $\boldsymbol{f}$ in the ionosphere is just the plasma-neutral collisional friction:

$$\boldsymbol{f} = -\nu_{in}\rho\left(\boldsymbol{V}-\boldsymbol{V}_n\right) \tag{28}$$

where $\boldsymbol{V}_n$ is the bulk flow velocity of the neutral atmosphere and ν_{in} is the ion-neutral collision frequency (electron collision effects on the plasma momentum equation being negligible over most of the ionosphere, except at the lowest altitudes). The expression for the part $\delta\boldsymbol{b}(0)_I$ contributed by the ionospheric term in the Dessler-Parker-Sckopke formula then is

$$\boldsymbol{\mu}\cdot\delta\boldsymbol{b}(0)_I = \int d\boldsymbol{r}\;\boldsymbol{f}\cdot\boldsymbol{r} = -\int d\boldsymbol{r}\;\nu_{in}\rho\left(\boldsymbol{V}-\boldsymbol{V}_n\right)\cdot\boldsymbol{r}\;, \tag{29}$$

proportional to the global integral of the collisional frictional force due to the differential *vertical* bulk flow of the plasma relative to the neutral atmosphere.

3.1 Role of the neutral atmosphere

One might be tempted to assume that the ionosphere is for the most part gravitationally bound and hence the vertical component of $\boldsymbol{f}$ must be small compared to gravity; this would imply an upper limit to $\boldsymbol{\mu}\cdot\delta\boldsymbol{b}(0)_I$ of order U_G, negligibly small in comparison to the contribution U_K of the plasma energy content in the magnetosphere. The vertical component, however, of the Lorentz force $\boldsymbol{J}\times\boldsymbol{B}/c$ for typical values of the auroral electrojet, whether up or down (up for eastward, down for westward electrojet), can easily be estimated and shown to exceed by some orders of magnitude the gravitational force $\rho\boldsymbol{g}$ on the plasma alone; it still is small, however, compared to $\rho^{(n)}\boldsymbol{g}$, the gravitational force on the neutral atmosphere. Clearly, the ionosphere is held in place by the neutral atmosphere, not by its own weight. The force $\boldsymbol{f}$ on the plasma is matched by an equal and opposite force on the neutral medium; that all the magnetic stresses on plasma in the ionosphere are transferred, via plasma-neutral collisions, entirely to the neutral atmosphere is in fact an essential assumption underlying the conventional ionospheric Ohm's law (Vasyliūnas and Song, 2005). The term $\int d\boldsymbol{r}\,\boldsymbol{f}\cdot\boldsymbol{r}$ in Eq. (29) also appears therefore, with opposite sign, in the virial theorem for the neutral atmosphere, the neutral-medium counterpart of Eq. (11):

$$-R_s{}^3\int d\Omega\,P^{(n)} = 2U_K^{(n)} + U_G^{(n)} - \int d\boldsymbol{r}\;\boldsymbol{f}\cdot\boldsymbol{r} \tag{30}$$

where the superscript $^{(n)}$ identifies quantities referring to the neutral atmosphere; the surface integrals have been evaluated under the assumption that at the outer boundary all surface terms are negligible and at the inner boundary only the pressure is significant (all the magnetic terms are of course absent).

With the use of Eq. (30), the ionospheric contribution to the Dessler-Parker-Sckopke formula may be expressed as

$$\boldsymbol{\mu} \cdot \delta \boldsymbol{b}(0)_I = \left({R_s}^3 \int d\Omega \, P^{(n)} \right) + 2U_K^{(n)} + U_G^{(n)} \,. \quad (31)$$

The right-hand side of Eq. (31) contains terms that are very large (by magnetospheric standards) but, being both positive and negative, very nearly cancel each other. If, for example, R_S is chosen to correspond to altitude 100 km where the atmospheric pressure is 0.3 dynes cm^{-2}, the contribution of the first term (effect of pressure at the inner boundary, numerically equal to twice the kinetic energy content within the volume *inward* of the boundary if the pressure there were held constant at its boundary value) alone would give a Dst value of -1.27 Gauss (-1.27×10^5 nT); it is balanced, of course, by $+1.27$ Gauss from the third term. The second term is smaller than the third (this is the condition that the atmosphere be gravitationally bound) by about the ratio (atmospheric scale height/Earth radius), but it would still contribute about -400 nT to Dst. In short, the terms on the right-hand side add up to very nearly zero, by the dynamics of the neutral atmosphere alone, and the magnetic effects represent only a very small perturbation on that. It is therefore more meaningful to rewrite Eq. (31) as

$$\boldsymbol{\mu} \cdot \delta \boldsymbol{b}(0)_I = \left({R_s}^3 \int d\Omega \, \delta P^{(n)} \right) + 2\delta U_K^{(n)} + \delta U_G^{(n)} \quad (32)$$

where the δ's are the changes of the various neutral energies that occur in association with magnetic disturbances. As these can be positive or negative, it is convenient to introduce a single symbol $\Delta U^{(n)}$ to represent the net change of the relevant energy content of the neutral atmosphere, allowing Eq. (32), the ionospheric contribution to Dst, to be finally expressed as

$$\boldsymbol{\mu} \cdot \delta \boldsymbol{b}(0)_I = \int d\boldsymbol{r} \, \boldsymbol{f} \cdot \boldsymbol{r} \;\approx\; \Delta U^{(n)} \,. \quad (33)$$

There does not seem to be any general method of evaluating $\Delta U^{(n)}$ other than integrating a complete global model for $\boldsymbol{f}\cdot\boldsymbol{r}$ (in which case $\boldsymbol{\mu}\cdot\delta\boldsymbol{b}(0)_I$ might as well be calculated directly from Eq. (27)). Some significant constraints on $\Delta U^{(n)}$ may, however, be imposed by considerations of energy transfer between the magnetosphere and the neutral atmosphere.

3.2 Magnetospheric perturbations of the neutral atmosphere

Two effects of magnetospheric processes on the neutral atmosphere are well known. The first is heating, both by direct heat flux from the magnetosphere (in the form of charged-particle precipitation) and by dissipation resulting from electrodynamic processes. The latter is commonly referred to as "ionospheric Joule heating" but Vasyliūnas and Song (2005) have shown that it is, for the most part, collisional frictional heating from the relative motion of plasma and neutrals; the conventionally used expression $\boldsymbol{J}\cdot\left(\boldsymbol{E}+\boldsymbol{V}^{(n)}\times\boldsymbol{B}/c\right)$ does, however, represent the total dissipation rate, about half of which goes into the neutral medium. The second process is acceleration of neutral flow by ion drag (e.g. Song et al., 2005, and references therein).

Neither one of these processes, however, is directly connected with the ionospheric effect on Dst. Heating increases the pressure and hence the kinetic energy content, but it also changes, through vertical displacements produced by the pressure imbalance, the gravitational potential energy content; the two come into equilibrium, satisfying Eq. (32) with zero on the left-hand side, on a time scale of a sound wave crossing a scale height (minutes or less). Ion drag affects the neutral atmosphere flow only on long time scales, an hour or more at typical ionospheric altitudes. Clearly, only those changes on the right-hand side of Eq. (32) that are *directly* related to the vertical component of the Lorentz force $\boldsymbol{J}\times\boldsymbol{B}/c$ in the ionosphere are relevant to Dst.

For the purpose of describing the effect of the vertical magnetic stresses exerted from the magnetosphere, the momentum equation for the neutral medium can be written in a form that explicitly includes the Lorentz force,

$$(\partial/\partial t)\left(\rho^{(n)}\boldsymbol{V}^{(n)}\right) + \nabla\cdot\left(\rho^{(n)}\boldsymbol{V}^{(n)}\boldsymbol{V}^{(n)}\right) + \nabla P^{(n)} = \rho^{(n)}\boldsymbol{g} + \boldsymbol{J}\times\boldsymbol{B}/c \,, \quad (34)$$

by noting that the neutral-ion collision term is equal to minus the ion-neutral collision term in the momentum equation for ionospheric plasma and hence to the Lorentz force which the latter balances. Taking the divergence of Eq. (34) and invoking the continuity equation gives a wave equation for the neutral density change $\delta\rho^{(n)}$, with $\nabla\cdot\boldsymbol{J}\times\boldsymbol{B}/c$ as the source term,

$$\left(\partial^2/\partial t^2\right)\delta\rho^{(n)} + \nabla\cdot\left({c_s}^2\nabla\delta\rho^{(n)}\right) - \boldsymbol{g}\cdot\nabla\delta\rho^{(n)} = \nabla\cdot\boldsymbol{J}\times\boldsymbol{B}/c \,, \quad (35)$$

where only terms linear in the perturbation amplitude have been kept; ${c_s}^2$ is the speed of sound. If $\delta\rho^{(n)}$ initially is zero everywhere and the Lorentz force turns on at $t=0$, the spatial-gradient terms are at first negligible and the density will initially change as $\sim t^2$,

$$\delta\rho^{(n)} \approx -\left(t^2/2\right)\nabla\cdot\boldsymbol{J}\times\boldsymbol{B}/c \approx \left(t^2/2\right)\boldsymbol{B}\cdot\nabla\times\boldsymbol{J}/c \approx \left(t^2/2\right)B_\theta\;(\partial/\partial r)\,J_\phi/c \quad (36)$$

(the second-order term $\boldsymbol{J}\cdot\nabla\times\boldsymbol{B}/c$ has been neglected); the second line, expressed in magnetic-dipole polar coordinates, follows from the fact that vertical variation of horizontal $\boldsymbol{J}$ is the dominant contribution to $\nabla\times\boldsymbol{J}$. Because the density is changing, the gravitational force also changes, coming into balance with the Lorentz force when

$$\delta\rho^{(n)}\ \boldsymbol{g}\cdot\hat{\boldsymbol{r}} \approx -\hat{\boldsymbol{r}}\cdot\boldsymbol{J}\times\boldsymbol{B}/c \approx B_\theta\ J_\phi/c \tag{37}$$

i.e. after a time τ given approximately by

$$\left(g\tau^2/2\right)\ (\partial/\partial r)\log|J_\phi| \equiv g\tau^2/2\lambda \sim \pm\, O\,(1)\ . \tag{38}$$

This is equal to the free-fall time across the typical vertical scale λ of the current density profile. The value of λ is determined primarily by the scale height of the neutral atmosphere (at altitudes above the maximum current density) and by the fall-off of the electron concentration (at altitudes below). Typically, $\lambda\sim$ some tens of kilometers, giving $\tau\sim$ minutes; inclusion of changes in pressure gradient in addition to gravitational force would modify this estimate somewhat, but not in order of magnitude.

We thus have a remarkable result: the neutral atmosphere, usually viewed as responding to magnetospheric plasma flows only on a time scale of hours, can nevertheless adjust itself in minutes in order to balance vertical magnetic stresses imposed from the magnetosphere. (This is possible because the requisite changes of the neutral density are very small compared to the density itself.) But it is precisely the global integral of these vertical stresses which represents the ionospheric contribution to Dst. Hence the primary contribution to $\Delta U^{(n)}$ is the change of gravitational potential energy in Eq. (32), given by the globally integrated, asymptotic long-time ($t\gg\tau$) limit of the density change described by Eq. (36). The changes of the pressure and kinetic-energy terms may, by comparison, be neglected: the associated adiabatic changes are smaller by the square of the ratio (sound speed/gravitational escape speed), while the non-adiabatic (heating) changes proceed independently and, as discussed above, establish their own gravitational equilibrium without magnetic effects.

3.3 Magnitude of the ionospheric contribution

The ionospheric term $\Delta U^{(n)}$ in the Dessler-Parker-Sckopke formula is thus equal to a fraction, not precisely specified but expected in general to be small, of the total energy supplied from the magnetosphere to the atmosphere. That energy, in turn, is comparable to the total energy supplied to the inner magnetosphere and the ring current, although the precise partitioning is a matter of some controversy (e.g. Weiss et al., 1992; Koskinen and Tanskanen, 2002; Feldstein et al., 2003, and references therein). In any case, $\Delta U^{(n)}$ should be small in comparison to the plasma kinetic energy term U_K and can for most purposes be neglected. (U_K includes the kinetic energy of all the plasma within the volume under consideration; no distinctions, e.g., between symmetric and partial ring currents or between ring current and other particle populations, are made.)

Note that the above result is an upper limit on the global average of the magnetic effects of the ionosphere and does not contain any information about the actual spatial configuration of these effects. The arguments by which it is derived presuppose vertical stress balance within the ionosphere; hence magnetic disturbances calculated from the asymmetric current system in some particular model are not necessarily consistent with the upper limit unless the model in question has also been constrained to satisfy stress balance.

4 Boundary surface terms

The outer surface that bounds the volume of integration, although described in Sect. 2 as "roughly the magnetopause," is in fact arbitrary (as pointed out also by Siscoe and Petschek, 1997), so far as any general arguments are concerned. The surface integrals in Eq. (25) represent the contribution to the disturbance field $\boldsymbol{b}(0)$ from everything that lies beyond the outer surface, whatever surface has been chosen. Contrary to what is sometimes supposed, they do not necessarily represent the effect of currents on the boundary (nothing so far requires the surface to coincide with particular currents), nor do they imply any specific assumptions about pressure in the region beyond the boundary. The values of the surface integrals depend of course on the choice of the outer boundary surface, but so do the values of the volume integrals U_K and U_b; the respective dependences must be such as to ensure that the disturbance field $\boldsymbol{b}(0)$, equal to the sum of all the terms on the right-hand side of Eq. (25), is independent of the choice of surface. The primary criterion, therefore, for selecting the outer boundary of the volume of integration is to make it as easy as possible to calculate the various individual terms of Eq. (25) – in particular, to evaluate the surface integrals conveniently, and to minimize the contribution of those volume terms (primarily U_b) that are difficult to calculate.

4.1 Choice of outer boundary

The magnetopause is a simple choice, especially convenient for evaluating the surface integral in the second line of Eq. (25): the integrand is the total (plasma + magnetic) pressure, which is the same on both sides of the magnetopause (hence it makes no difference if the surface is chosen just inside or just outside the magnetopause) and can be related to solar wind parameters and the shape of the magnetopause by the Newtonian approximation (e.g. Spreiter et al., 1968)

$$P_{\text{total}} \equiv P + \left(B^2/8\pi\right) \simeq \kappa\rho_{sw}\left(\boldsymbol{V}_{sw}\cdot\hat{\boldsymbol{n}}\right)^2 + P_{sw} \tag{39}$$

where $\hat{\boldsymbol{n}}$ is the unit vector normal to the magnetopause surface, and the dimensionless constant $\kappa\approx 0.9$. Less simple is the inclusion of the magnetotail and the associated need to evaluate the surface integral in the fourth line of Eq. (25). It is possible (e.g. Siscoe and Petschek, 1997) to choose the outer boundary surface coincident with the magnetopause everywhere, including the magnetotail, out to some indefinitely distant closing cross-section far downstream of the Earth (to avoid having to include the surface integral in the fifth line of Eq. (25), the surface must now be chosen just inside the magnetopause). With this choice, however, there is a large contribution to U_b from the magnetic energy in the magnetotail, proportional to the (indefinitely large) effective length $\mathcal{L}_{MT}$ of the magnetotail. The surface integral in the fourth line of Eq. (25) is also proportional to $\mathcal{L}_{MT}$, as well as to the open magnetic flux Φ_{MT}, and in fact it largely cancels the magnetotail contribution to U_b; this can be demonstrated either by direct calculation from a simple model (with pressure balance within the magnetotail taken into account) or, more elegantly but subtly, by noting that the entire derivation of Eq. (25) can be redone for a volume enclosing only the magnetotail and excluding the Earth – then the left-hand side is zero, and all the magnetotail terms add up to zero.

To avoid having to calculate large terms that then nearly cancel each other, the boundary surface should be chosen to exclude as much of the magnetotail as possible. The following seems to be a simple but adequate configuration: the volume of integration is bounded on the nightside by a plane perpendicular to the Sun-Earth line and located a distance X antisunward of the Earth; everywhere sunward of that, the volume is bounded by the magnetopause. The distance X is chosen to correspond to the earthward edge of the magnetotail near midnight, a reasonable criterion being

$$\mu X^3 \approx B_T \qquad (40)$$

with μ the Earth's dipole moment and B_T the magnetic field of the near-Earth magnetotail; typically $B_T \approx 30$ nT and $X \approx 10\, R_E$.

In Sects. 4.2 and 4.3 I estimate the boundary surface contributions to the Dessler-Parker-Sckopke formula on the basis of this simple configuration. It is convenient to separate them into pressure effects (second and third lines of Eq. (25)), present even if there were no magnetotail, and effects specifically associated with the magnetotail (fourth line of Eq. 25); I neglect the fifth line of Eq. (25) as well as all pressure anisotropy effects. In some treatments, boundary surface contributions appear implicitly, without explicit introduction of a boundary; I discuss some examples in Appendix A.

4.2 Total pressure (Chapman-Ferraro) integral

The contribution of the surface integrals related to pressure is given by

$$\begin{aligned}\boldsymbol{\mu} \cdot \delta \boldsymbol{b}(0)_{CF} = &-\int d\boldsymbol{S} \cdot \boldsymbol{r} \left[P + \left(B^2/8\pi \right) \right] \\ &+ \int d\boldsymbol{S} \cdot \left[\boldsymbol{B}_d \times (\boldsymbol{r} \times \boldsymbol{B}_d) \right] / 8\pi \,. \end{aligned} \qquad (41)$$

In the first line of Eq. (41), the integrand (total pressure) may be taken as given by Eq. (39) for the integral over the magnetopause and as ${B_T}^2/8\pi$ for the integral over the nightside plane; ${B_T}^2/8\pi$ is in general also proportional to the solar wind dynamic pressure $\rho_{sw} {V_{sw}}^2$ (the solar wind thermal pressure P_{sw} may be assumed negligible in this context). It is convenient to parametrize the linear size of the magnetosphere by the distance to the subsolar magnetopause R_{MP} and to introduce the nominal Chapman-Ferraro distance R_{CF} defined by

$$\mu / {R_{CF}}^3 = \left(8\pi \rho_{sw} {V_{sw}}^2 \right)^{1/2} . \qquad (42)$$

Equation (41) may then be written as

$$\boldsymbol{\mu} \cdot \delta \boldsymbol{b}(0)_{CF} = -\mu \zeta \left(8\pi \rho_{sw} {V_{sw}}^2 \right)^{1/2} \qquad (43)$$

where ζ is a dimensionless number given by

$$\zeta \equiv \left[\sigma_1 \left(R_{MP}/R_{CF} \right)^3 - \sigma_2 \left(R_{CF}/R_{MP} \right)^3 \right] \qquad (44)$$

and σ_1, σ_2 are numerical factors dependent on the shape of the magnetopause surface (σ_1 depends also on the ratio ${\beta_T}^{-1} \equiv {B_T}^2/8\pi \rho_{sw} {V_{sw}}^2$). If the plasma pressure just inside the subsolar magnetopause is negligible, $(R_{MP}/R_{CF})^3$ equals the ratio of actual to dipole magnetic field magnitude there, usually assumed to lie between 2 (planar magnetopause) and 3 (concentric spherical magnetopause).

As a very simple illustrative example, assume the subsolar magnetopause to be as far sunward as the earthward edge of the magnetotail is antisunward ($R_{MP}=X$) and take the magnetopause to be a hemispherical surface of radius $2R_{MP}$ (hence with center at distance X antisunward of the Earth). All the integrals can then be calculated analytically, with the results

$$\sigma_1 = 0.42 + 0.5 {\beta_T}^{-1} \qquad \sigma_2 = 0.73 \,. \qquad (45)$$

The contribution to $\boldsymbol{b}(0)$ given by Eq. (43) is identical in form to the well-known "pressure correction" to Dst, usually obtained empirically. The derivation given here relates it explicitly to the size and shape of the magnetopause. The dependence, displayed in Eq. (44), on the ratio R_{MP}/R_{CF} (i.e., the actual subsolar distance compared to the scale length determined by solar wind dynamic pressure alone) may be of particular interest in connection with the reported decrease of the pressure correction with increasing southward interplanetary magnetic field (McPherron and O'Brien, 2001; Siscoe et al., 2005b).

4.3 Magnetic flux (magnetotail) integral

The contribution of the surface integrals related to the magnetic field normal to the boundary surface is given by

$$\boldsymbol{\mu} \cdot \delta \boldsymbol{b}(0)_{MT} = \int d\boldsymbol{S} \cdot \boldsymbol{B} \left[\boldsymbol{B} - (\boldsymbol{B}_d/2) \right] \cdot \boldsymbol{r} / 4\pi \,. \qquad (46)$$

The magnetic field normal to the dayside magnetopause is generally considered to be very small, the bulk of the open magnetic flux coming through the magnetotail. Hence for computing the integral in Eq. (46) only the nightside plane need to be considered. One may write the magnetic flux through the magnetotail as

$$\Phi_{MT} = \int d\boldsymbol{S} \cdot \boldsymbol{B} \tag{47}$$

where the integral is taken only over *one half* of the cross-section surface, where $\boldsymbol{B}\cdot\hat{\boldsymbol{x}}$ has one sign (subsequent equations have therefore been multiplied by a factor 2). Then the integral in Eq. (46) can be written as

$$\Phi_{MT} \left\langle [\boldsymbol{B} - (\boldsymbol{B}_d/2)] \cdot \boldsymbol{r}/2\pi \right\rangle$$

where $\langle\ \rangle$ denotes the average over the surface. With $\boldsymbol{B}$ very nearly tail-like on the nightside surface, the result is

$$\boldsymbol{\mu} \cdot \delta\boldsymbol{b}(0)_{MT} = \Phi_{MT} \left\langle B_T X \right\rangle (1-\delta)/2\pi \tag{48}$$

where the second term δ, obtained from $\langle \boldsymbol{B}_d \cdot \boldsymbol{r} \rangle$, is

$$\delta = (30\,\mathrm{nT}/B_T)\,(X/10\,R_E)^2\, F\,(R_T/X) \tag{49}$$

with R_T the radius of the magnetotail at the distance X (R_T=$2R_{MP}$ in the simple model discussed in Sect. 4.2) and F the function defined as

$$F\,(u) \equiv \frac{4}{\pi u^2}\left[\log\left(u+\sqrt{1+u^2}\right) - \frac{u}{\sqrt{1+u^2}}\right]. \tag{50}$$

It is easily shown that the maximum value of $F(u)$ is $F(1.2)$=0.22; hence δ is a small correction term which can usually be neglected.

With δ neglected and and with the open magnetic flux approximated as

$$\Phi_{MT} \approx (1/2)\,\pi {R_T}^2 B_T \tag{51}$$

Eq. (48) may be written analogously to Eq. (41) as

$$\boldsymbol{\mu} \cdot \delta\boldsymbol{b}(0)_{MT} = +\mu\varsigma\left(8\pi\rho_{sw} {V_{sw}}^2\right)^{1/2} \tag{52}$$

where the dimensionless number ς is given by

$$\varsigma \equiv {\beta_T}^{-1}\left({R_T}^2 X/4{R_{CF}}^3\right) \tag{53}$$
$$= {\beta_T}^{-1}\,(R_{MP}/R_{CF})^3\left[(R_T/2R_{MP})^2\,(X/R_{MP})\right].$$

For the simple model discussed in Sect. 4.2, the factor in [ ] in the second line of Eq. (53) equals unity.

The contribution to $\boldsymbol{b}(0)$ given by Eq. (48) is similar in form and value to previous estimates derived from various models of magnetotail currents (Alexeev et al., 1996; Arykov and Maltsev, 1996) or from empirical arguments (Ostapenko and Maltsev, 1998, 2000; Turner et al., 2000; Ohtani et al., 2001). As derived here, however, it does not depend on specific assumptions about current systems and is not affected by controversies (e.g. Maltsev and Ostapenko, 2002; Turner et al., 2002) on what constitutes the magnetotail current. Contrary to what sometimes seems to be supposed, the magnetotail contribution in general cannot be expressed as simply a stated fraction of the total disturbance field; the relation between Φ_{MT} and U_K is complex, indirect, and strongly dependent on time history (Vasyliūnas, 2006, and references therein). Historical note: before Dessler and Parker (1959) had shown that a southward disturbance field at the Earth could be related to the energy content of trapped plasma, the entire storm-time depression of the geomagnetic field was sometimes ascribed to the magnetotail (Parker, 1958; Piddington, 1960).

The magnetic energy content within the magnetotail can be expressed in a form resembling the right-hand side of Eq. (48):

$$U_{MT} = \int d\boldsymbol{r}\; B^2/8\pi \;\approx\; \Phi_{MT} \left\langle B_T \mathcal{L}_{MT} \right\rangle/4\pi \tag{54}$$

where the volume integral is over the magnetotail (i.e., anti-sunward of the plane surface at X), and $\mathcal{L}_{MT}$ is the effective length of the magnetotail; note that $\mathcal{L}_{MT}$, defined here on the basis of energy content, is in general not the same as the length of the magnetotail defined by Dungey (1965) on the basis of plasma flow time across the open field region. The contribution to $\boldsymbol{\mu}\cdot\boldsymbol{b}(0)$ of the magnetotail boundary surface integral may thus be expressed as the fraction $2X/\mathcal{L}_{MT}$ of the total magnetic energy of the magnetotail, or equivalently as the magnetic energy within a near-Earth segment of the magnetotail of length $2X$.

5 The formula for *Dst*

The final, generalized form of the Dessler-Parker-Sckopke theorem can be written as

$$\boldsymbol{\mu} \cdot \boldsymbol{b}(0) = 2U_K + U_b + U_I \tag{55}$$
$$-\,\mu\zeta\left(8\pi\rho_{sw} {V_{sw}}^2\right)^{1/2} + \Phi_{MT}\left\langle B_T X\right\rangle(1-\delta)/2\pi\,.$$

Here U_I=$\Delta U^{(n)}$+U_G represents the combined ionospheric and gravitational contributions, and the other symbols have already been defined. Note from Eq. (24) that $\boldsymbol{b}(0)$ is actually calculated as the average of $\boldsymbol{b}$ over the surface of a sphere and is expressed as the field at the origin (equal to the average because $\boldsymbol{b}$ is assumed curl-free within the sphere and hence satisfies $\nabla^2\boldsymbol{b}$=0) purely for convenience. As a consequence, $\boldsymbol{b}(0)$ is not affected by the magnetic field of the shielding currents within the Earth that keep out the time-varying external magnetic fields – any Cartesian component of an internal magnetic field vanishes when globally averaged.

The observed Dst differs from $\boldsymbol{b}(0)$ in two respects. First, as already noted, Dst<0 corresponds to $\boldsymbol{\mu}\cdot\boldsymbol{b}(0)$>0. Second, and more important, Dst is derived not from a global average but only from a local-time average near the equator, which does not eliminate contributions from internal fields. Thus, to first approximation at least, Dst is equal to $-\boldsymbol{b}(0)$ multiplied by a factor $\xi \approx 3/2$ to include the effect of the shielding currents. Equation (55), the Dessler-Parker-Sckopke theorem, then provides the following formula for Dst:

$$Dst = -\left(2U_K + U_b + U_I\right)(\xi/\mu) \qquad (56)$$
$$+ \xi\zeta \left(8\pi \rho_{sw} V_{sw}{}^2\right)^{1/2} - \Phi_{MT} \langle B_T X \rangle (1-\delta)\,\xi/2\pi\mu\,.$$

In Eqs. (55) and (56), the first line contains the energy terms: (twice) the kinetic energy of the plasma, (once) the energy of the magnetic perturbation field, and the ionospheric contribution. The second line of both equations contains the two boundary surface terms: the magnetopause contribution, dependent primarily on the square root of the solar-wind dynamic pressure, and the magnetotail contribution, proportional to the open magnetic flux.

When applying the Dessler-Parker-Sckopke formula to analyse and model Dst in relation to solar-wind parameters for particular geomagnetic storms, Eq. (56) may be compared to that used in most previous studies (e.g. Burton et al., 1975; O'Brien and McPherron, 2000, 2002; McPherron and O'Brien, 2001; Liemohn et al., 2001, 2002; Jordanova et al., 1998, 2003; Siscoe et al., 2005a,b):

$$Dst = -2U_K\,(\xi/\mu) + (\text{const.}) \left(8\pi \rho_{sw} V_{sw}{}^2\right)^{1/2} \qquad (57)$$

in the notation of this paper. Equation (56) differs from Eq. (57) by the addition of the magnetotail term (considered occasionally in previous work but not systematically included) and by the precise definition of the magnetopause term, furthermore by the addition of the ionospheric contribution which is important conceptually but not practically. The difference in the predicted numerical values of Dst is in most cases probably not very large; there may, however, be significant differences of interpretation, particularly when time derivatives are considered (Vasyliūnas, 2006).

6 Conclusions

Fundamentally, the Dessler-Parker-Sckopke theorem is a global stress balance condition: it describes the amount by which the magnetic dipole field (anchored in the massive Earth and held in place ultimately by its gravity) must be deformed near the Earth in order for the magnetosphere to remain in place, neither flying apart nor collapsing under the action of all the energies it contains. The ionospheric contribution to the deformation of the field arises when the magnetic field must support a vertical imbalance between gravity and pressure. The resulting disturbance field is proportional to the corresponding difference between the kinetic and the gravitational potential energy. Since these energy changes are produced by energy input to the ionosphere and atmosphere from the magnetosphere, and since only a fraction of the input goes into a non-equilibrium partitioning between kinetic and gravitational energies, the ionospheric contribution is in general small compared to the direct effect of magnetospheric plasma; its modification of Dst is at most comparable to other uncertain terms (such as the effect of energy in the disturbance field), or even to the uncertainties implied by the difference between the theoretical and the observational definitions of Dst. The deformation of the field by stresses external to the magnetosphere appears in the theorem as boundary surface integrals which can be calculated from the boundary shape and the external parameters.

Appendix A

Implicit surface terms

Throughout the paper it has been taken for granted that there is a specified outer boundary of the volume of integration, with the values of the quantities appearing in the surface integrals explicitly given. Sometimes, however, the volume integral U_K is calculated over a finite volume (typically out to a maximum radial distance or a maximum magnetic shell parameter L) without any explicit discussion of outer boundaries or surface integrals; one may, for instance, be dealing with simulation results calculated only over a finite numerical domain, or else considering the contribution to the disturbance field only by plasma within the inner magnetosphere. Liemohn (2003) describes several such cases, evaluating the disturbance field both from the Dessler-Parker-Sckopke theorem in its original form Eq. (1) and from the Biot-Savart law; he concludes that the theorem overpredicts the perturbation, and that the "true" value can be recovered by including a correction to remove what he calls a truncation current term.

The discrepancies and corrections described by Liemohn (2003) can in fact be completely understood on the basis of the boundary surface terms, in a way that highlights the difference (mentioned in Sect. 2) between the Biot-Savart law and the momentum equation as the actual physical basis of the Dessler-Parker-Sckopke theorem. Mathematically, the following three procedures for calculating the right-hand side of the Dessler-Parker-Sckopke formula are completely equivalent: (a) calculate the volume integrals out to $L{=}L_b$ and then stop, (b) calculate the volume integrals out to infinity (nominally) but assume that for $L{>}L_b$ the integrands (e.g., pressure) are zero, (c) choose the outer boundary surface at $L{=}L_b$, calculate the volume integrals, and set the surface integrals to zero. Liemohn (2003) realizes the equivalence of procedures (a) and (b) and points out, correctly, that the implied drop of pressure to zero requires a balancing Lorentz force in the momentum equation at $L{=}L_b$ and that the Dessler-Parker-Sckopke formula, being based on the momentum equation, includes the magnetic effects of the associated ("truncation") current. Since the actual pressure in the magnetosphere usually does not drop to zero at $L{=}L_b$, he regards this truncation-current contribution as an error in the Dessler-Parker-Sckopke formula and proposes a method of correction.

For a simple model distribution of pressure given by

$$\begin{aligned} P &= (L - L_m)\exp(-L/\Delta L) \qquad L_m \le L \le L_b \\ &= 0 \qquad L \le L_m \end{aligned} \tag{A1}$$

(times a normalizing constant which plays no role in the calculation), Liemohn (2003) evaluates the disturbance field at the origin by two methods: b_1 from Eq. (1), with U_K obtained by integrating the energy density out to $L{=}L_b$, and b_2 from the Biot-Savart integral over the currents implied by the pressure gradient, carried out to some distance *beyond* $L{=}L_b$ with non-zero (continuous and adiabatically decreasing) pressure; b_1 thus contains the truncation-current contribution but b_2 does not. (My notation differs from that of Liemohn, 2003, and in particular my $\{b_1, b_2\}\equiv$ his $\{\Delta B_{DPS}, \Delta B_{BSI}\}$). Defining the disturbance ratio $\mathcal{B}\equiv b_2/b_1$ and the pressure ratio $\mathcal{P}\equiv(P_p/P_b)$, where $P_p=$ peak pressure (occurring at $L_p{=}L_m{+}\Delta L$) and $P_b=$ pressure at the boundary ($L{=}L_b$), he finds that, for a wide range of model parameters, there is a nearly one-to-one relation between $\mathcal{B}$ and $\mathcal{P}$. Given the pressure profile, this relation can be used to infer the value of the disturbance ratio $\mathcal{B}$, the difference $(1{-}\mathcal{B})$ being regarded as a measure of the truncation error.

A physically more meaningful approach, however, is provided by procedure (c). The surface defined by $L{=}L_b$ is, de facto, the outer boundary surface, and the surface integrals *must* be specified if the Dessler-Parker-Sckopke theorem is to be applied. The two different estimates of the disturbance field, b_1 and b_2, correspond, in reality, to two different specifications of the surface integrals (in the present context of a linear treatment, with assumed dipolar field in the magnetosphere, only the pressure terms are significant, all the magnetic field terms summing to zero): b_1 presumes zero pressure at the boundary, whereas b_2 presumes boundary pressure equal to the adjacent interior pressure. The disturbance ratio is thus given by

$$\mathcal{B} \equiv \frac{2U_K - \int d\boldsymbol{S}\cdot\boldsymbol{r}P}{2U_K}\,. \tag{A2}$$

For the model of Eq. (A1), the pressure ratio is

$$\mathcal{P} = e^x/(x+1) \qquad x \equiv \left(L_b - L_p\right)/\Delta L\,. \tag{A3}$$

In the following I assume, for simplicity, isotropic pressure with filled loss cone (Liemohn, 2003, assumes empty loss cone and considers both isotropic and anisotropic pressures). Then P depends only on L, and $\int d\boldsymbol{S}{\cdot}\boldsymbol{r}P{=}P_b\int d\boldsymbol{S}{\cdot}\boldsymbol{r}{=}3P_b\int d\boldsymbol{r}$ or

$$\int d\boldsymbol{S}\cdot\boldsymbol{r}P = P_b\int_0^{L_b} L^2\,dL\,, \quad 2U_K = 3\int_0^{L_b} L^2\,dL\,P \tag{A4}$$

(the additional integrations over angles, common to both terms, are not shown). From Eqs. (A1), (A2), (A3), and (A4), the disturbance ratio can be calculated in closed form:

$$\begin{aligned} \mathcal{B} &= 1 - [(x+1)/3I\delta] \\ I &\equiv e^{x+1}\left[p^2 + 2p\delta + 3\delta^2\right] \\ &\quad - \left[x\left(1+2\delta+2\delta^2\right) + 2\left(1+3\delta+4\delta^2\right)\right] \end{aligned} \tag{A5}$$

where $\delta \equiv \Delta L/L_b \quad p \equiv L_p/L_b{=}1-x\delta$.

Table A1. Relation between pressure and disturbance ratios.

	$\mathcal{B}$			
$\mathcal{P}$	(Liemohn, 2003)	($L_p = 4.5$	$= 3.5$	$= 2.5$)
1.277	0	0.0706	0.1291	0.1285
3.825	0.5	0.4727	0.5257	0.5110
10.0	0.7355	0.7204	0.7428	0.7242
41.00	0.9	0.9015	0.9053	0.8934

In Table A1, the pressure and disturbance ratios given by Eqs. (A3) and (A5), with $L_b{=}6.5$ held fixed and L_p varied, are compared with the corresponding values from the model of Liemohn (2003). The first column contains the assumed $\mathcal{P}$ values and the second column the $\mathcal{B}$ values from Table 1 (isotropic case only) of Liemohn (2003). The remaining columns contain the $\mathcal{B}$ values calculated here for each assumed $\mathcal{P}$ with various L_p, solving Eq. (A3) numerically for x as a function of $\mathcal{P}$, determining ΔL from x, and inserting into Eq. (A5). It is evident that the agreement is sufficiently good (particularly in view of the different assumptions concerning the loss cone) to confirm the reinterpretation proposed here: removing the supposed truncation current term is nothing else than merely including the surface term that necessarily is present if the volume is bounded and the pressure at the boundary is not zero.

Alternatively, if use of a bounded volume is viewed as a way of isolating the contribution from that volume to the total disturbance field $\boldsymbol{b}(0)$, the expressions given by the Dessler-Parker-Sckopke theorem and by the Biot-Savart integral need not give the same result. The equivalence of the two expressions holds only for the complete integrals and not for the integrands (this is particularly apparent from the proof of the equivalence in Vasyliūnas, 2001); when only partial contributions are evaluated, the so-called truncation current term is the difference.

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